HOMEWORK
From Dieter
2-1, 2-2, 2-5

Module #2

Transformation of stresses in 2-D

READING LIST

DIETER: Ch. 2, Pages 20-27

Ch. 2 in McClintock and Argon
Ch. 7 in Edelglass

Objectives

- Develop equations for transformation of axes.
- Apply equations to determine principal normal and shear stresses.
- Perform transformation graphically using Mohr’s circle.

Why bother?

- Engineering structures are subjected to many different types of applied loads:
  - Tension
  - Compression
  - Bending
  - Torsion
  - Pressure
  - Combinations of the above
- Complex states of normal and shear stress occur.

Why bother?

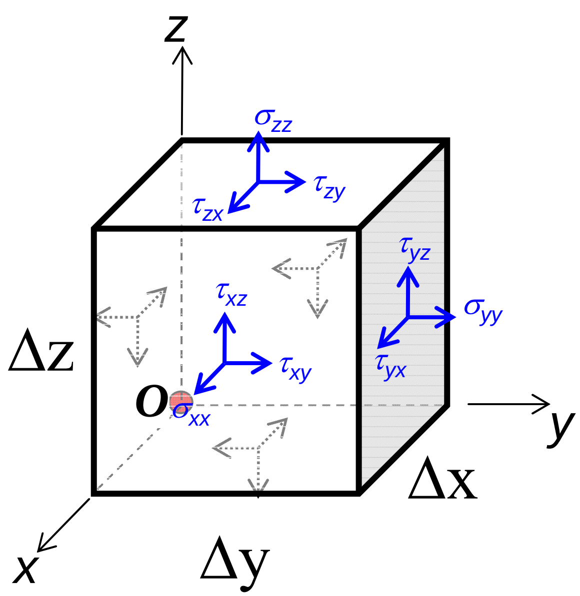
- These stress components vary in magnitude and direction relative to their location in the structure and the coordinate system with which they are related to.
- As engineers we need to make sure that the structures we design don't fail as a result of the applied stresses.
- Thus, we need to identify locations where stresses are the most severe. Then we can do more detailed analysis if needed.

Principal Stresses and Axes

- Principal stresses:
  - The most severe stresses (i.e., the maximum and minimum stresses) in a stress state.

- Principal axes:
  - Directions in which principal stresses act.

The Stress Tensor



$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

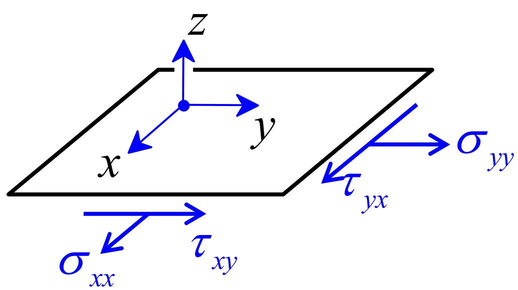
If we limit ourselves to two dimensions, we achieve a condition called plane stress.

This is where all stresses in one dimension become zero.
[see the next page]

PLANE STRESS

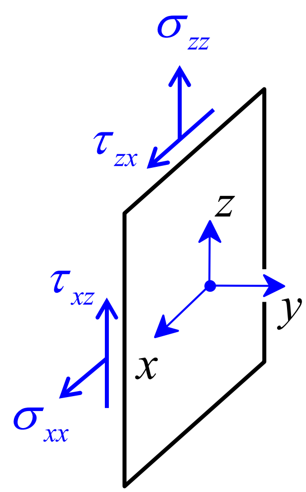
All stresses are zero in one primary direction

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \sigma_{xx} & 0 & \tau_{xz} \\ 0 & 0 & 0 \\ \tau_{zx} & 0 & \sigma_{zz} \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{yy} & \tau_{yz} \\ 0 & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$



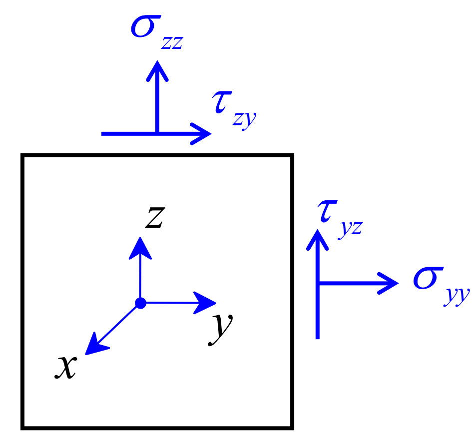
$\tau_{xy} = \tau_{yx}$

(a)



$\tau_{xz} = \tau_{zx}$

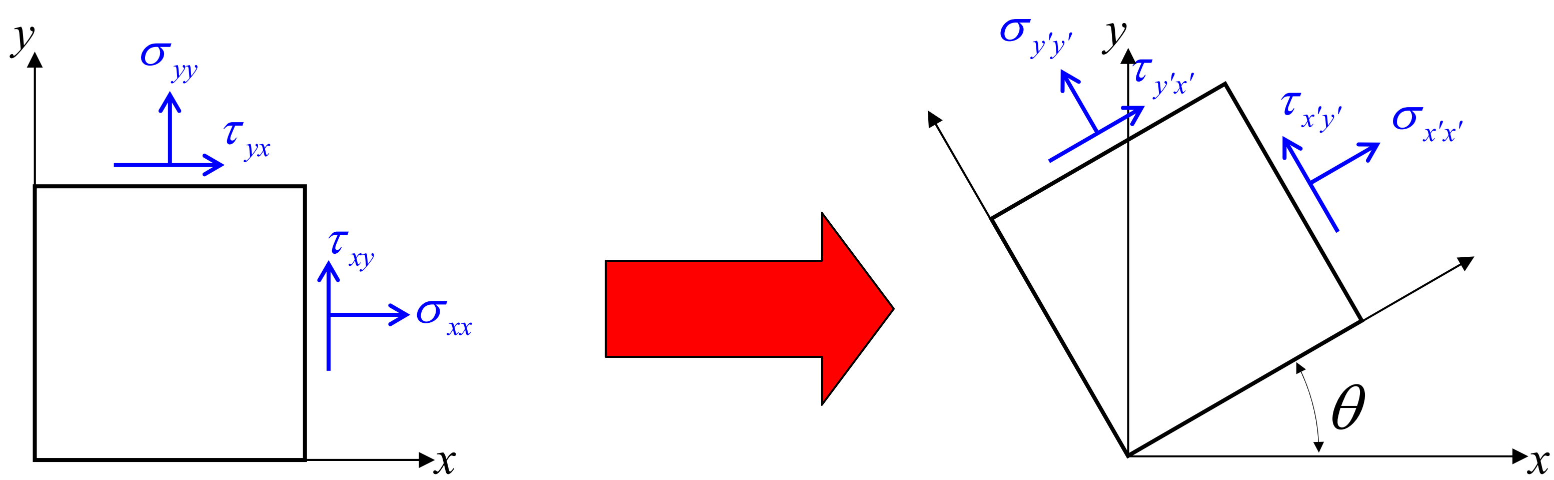
(b)



$\tau_{yz} = \tau_{zy}$

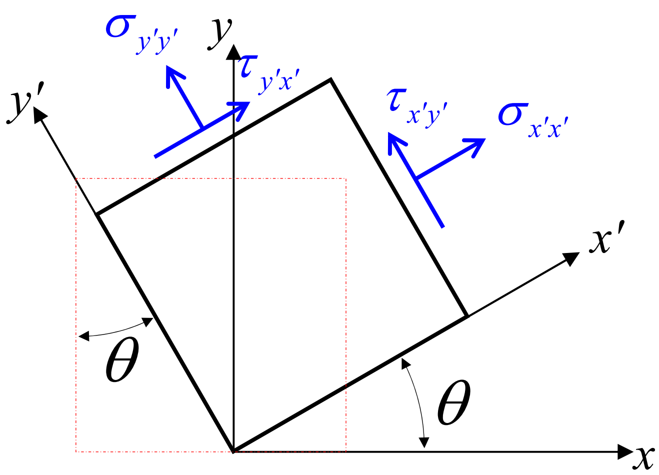
(c)

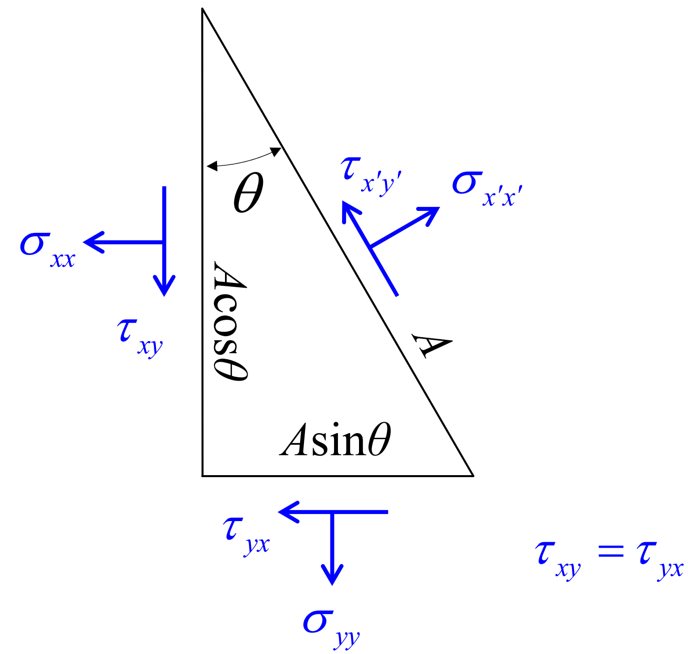
$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} & 0 \\ \tau_{x'y'} & \sigma_{y'y'} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Transformation is accomplished via a force balance

Let's first balance forces parallel to x' followed by y'





$$\sum F_{x'} = 0 = \sigma_{x'x'} A - \tau_{yx}\left(A\sin\theta\right)\cos\theta - \tau_{xy}\left(A\cos\theta\right)\sin\theta$$
$$-\sigma_{xx}\left(A\cos\theta\right)\cos\theta - \sigma_{yy}\left(A\sin\theta\right)\sin\theta$$

$$\sum F_{y'} = 0 = \tau_{x'y'} A - \tau_{xy}\left(A\cos\theta\right)\cos\theta + \tau_{yx}\left(A\sin\theta\right)\sin\theta$$
$$+\sigma_{xx}\left(A\cos\theta\right)\sin\theta - \sigma_{yy}\left(A\sin\theta\right)\cos\theta$$

Solving for $\sigma_{x'x'}$ and $\tau_{x'y'}$ yields:

$$\boxed{\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta}$$

$$\boxed{\tau_{x'y'} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)}$$

Since $\sigma_{y'y'}$ is 90° away from $\sigma_{x'x'}$, $\sigma_{y'y'}$ is:

$$\sigma_{y'y'} = \sigma_{xx}\cos^2\left(\theta + \frac{\pi}{2}\right) + \sigma_{yy}\sin^2\left(\theta + \frac{\pi}{2}\right) + 2\tau_{xy}\sin\left(\theta + \frac{\pi}{2}\right)\cos\left(\theta + \frac{\pi}{2}\right)$$

$$\boxed{\sigma_{y'y'} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta}$$

THESE THREE BOXED EQUATIONS ARE KNOWN AS THE
TRANSFORMATION OF STRESS EQUATIONS

We can simplify the Transformation of Stress equations by invoking the double-angle identities

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$2\sin\theta\cos\theta = \sin 2\theta$$

Transformation of Stress Equations

(for plane strain)

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = \frac{\sigma_{yy} - \sigma_{xx}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

They provide us with the stresses re-oriented to a new coordinate system

$$\begin{matrix} \sigma_{x'x'} & \tau_{x'y'} & 0 \\ \tau_{x'y'} & \sigma_{y'y'} & 0 \\ 0 & 0 & 0 \end{matrix}$$

An interesting observation

$$\sigma_{x'x'} + \sigma_{y'y'} = \begin{pmatrix} \dfrac{\sigma_{xx}+\sigma_{yy}}{2} + \dfrac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ + \\ \dfrac{\sigma_{xx}+\sigma_{yy}}{2} - \dfrac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \end{pmatrix} = \sigma_{xx} + \sigma_{yy}$$

$$\boxed{\sigma_{x'x'} + \sigma_{y'y'} = \sigma_{xx} + \sigma_{yy}}$$

↑

This quantity is invariant! It doesn't change regardless of the coordinate system.

2 stress invariants for plane strain

- I_1 is the sum of the main diagonal of the stress tensor.

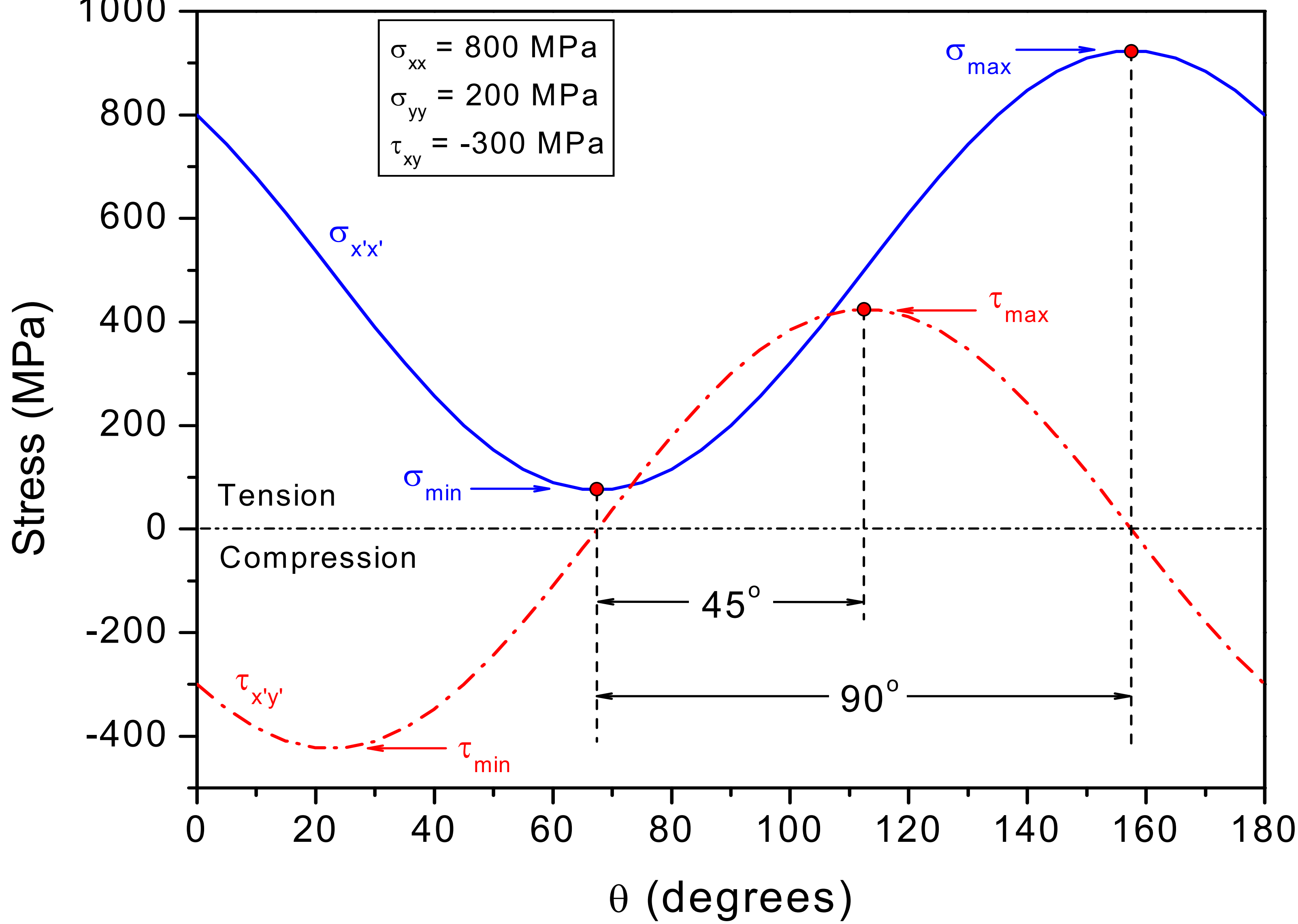
$$I_1 = \sigma_{xx} + \sigma_{yy} = \sigma_{x'x'} + \sigma_{y'y'} = \cdots$$

- I_2 is the sum of the principal minors.

$$I_2 = \sigma_{xx}\sigma_{yy} - \tau_{xy}^2 = \sigma_{x'x'}\sigma_{y'y'} - \tau_{x'y'}^2 = \cdots$$

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = \frac{\sigma_{yy} - \sigma_{xx}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$



Principal planes

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

from previous viewgraph, normal stress is maximum (or minimum) when:

$$\frac{d\sigma_{x'x'}}{d\theta} = 0 = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\left(2\sin 2\theta\right) + 2\tau_{xy}\cos 2\theta$$

Solving for θ yields the plane where maximum and minimum normal stress occur.

$$\tan 2\theta_{principal} = \frac{\tau_{xy}}{\left(\sigma_{xx} - \sigma_{yy}\right)/2}$$

By definition this is also the plane where shear stress vanishes

Maximum in-plane shear stress

$$\tau_{x'y'} = \frac{\sigma_{yy} - \sigma_{xx}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Shear stress is maximum/minimum when:

$$\frac{d\tau_{x'y'}}{d\theta} = 0 = -\frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta - 2\tau_{xy} \sin 2\theta$$

Solving for θ yields the plane of maximum/minimum shear stress

$$\tan 2\theta_{shear} = -\frac{\left(\sigma_{xx} - \sigma_{yy}\right)/2}{\tau_{xy}}$$

An average normal stress is superimposed on these planes

$$\sigma_{average} = \frac{\sigma_{xx} + \sigma_{yy}}{2}$$

MAXIMUM & MINUMUM PRINCIPAL STRESSES FOR A 2-D STATE OF STRESS

$$\begin{matrix}\sigma_{\max} \\ \sigma_{\min}\end{matrix} = \begin{matrix}\sigma_1 \\ \sigma_2\end{matrix} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

5 minute break

Mohr's Circle for Stress

- Developed in 1882; a graphical way to represent the transformation of stress equations.

$$\sigma_{x'x'} = \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right) + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = \left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

- Re-write the equations as:

$$\sigma_{x'x'} - \sigma_{\text{average}} = \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = \left(\frac{\sigma_{yy} - \sigma_{xx}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Mohr's Circle

- Square both sides of each equation and add them together, which yields:

$$\left(\sigma_{x'x'} - \sigma_{\text{average}}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2$$

or

$$\left(\sigma_{x'x'} - \sigma_{\text{average}}\right)^2 + \tau_{x'y'}^2 = R^2$$

- Equation for a circle expressed in (σ, τ) coordinates with a center at $(\sigma_{\text{average}}, 0)$.

Mohr's Circle – cont'd

- Recall the equation for a circle:

  $$(x-h)^2 + y^2 = R^2$$

  where h is the location of the center on the x axis (i.e., the center of the circle), and R is its radius.

- The center of Mohr's circle lies at ($\sigma_{average}$,0). $\sigma_{\text{average}} = \left(\dfrac{\sigma_{xx} + \sigma_{yy}}{2}\right)$

- The radius of Mohr's circle is:

  $$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \tau_{\max} = \tau_3$$

- $\sigma_{max} = \sigma_1 = \sigma_{average} + R$

- $\sigma_{min} = \sigma_2 = \sigma_{average} - R$

The steps for construction are provided on the next 2 pages of the handout <u>and/or</u> in your mechanics of materials book.

Steps in Construction of Mohr's Circle

1. Show the stresses σ_{xx}, σ_{yy}, and τ_{xy} on a cube. Label the vertical plane V and the horizontal plane H.

2. Write the coordinates of points V and H as V(σ_{xx}, -τ_{xy}) and H(σ_{yy}, τ_{yx}). A positive value for τ_{ij} produces a CW moment about the center of the cube (i.e. CW rotation of the cube).

3. Draw the horizontal axis with the tensile normal stress to the right (i.e., positive) and the compressive normal stress to the left (i.e., negative). Draw the vertical axis with the clockwise (CW) direction of shear stress (i.e., positive) up and the counterclockwise (CCW) direction of rotation down.

4. Locate points V and H and join the points by drawing a line. Label the point where line VH intersects the horizontal axis as C, the center of the circle. The center has coordinates C($\sigma_{average}$, 0).

5. Draw Mohr's circle with point C as the center and a radius, R of

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \tau_{\max} = \tau_3$$

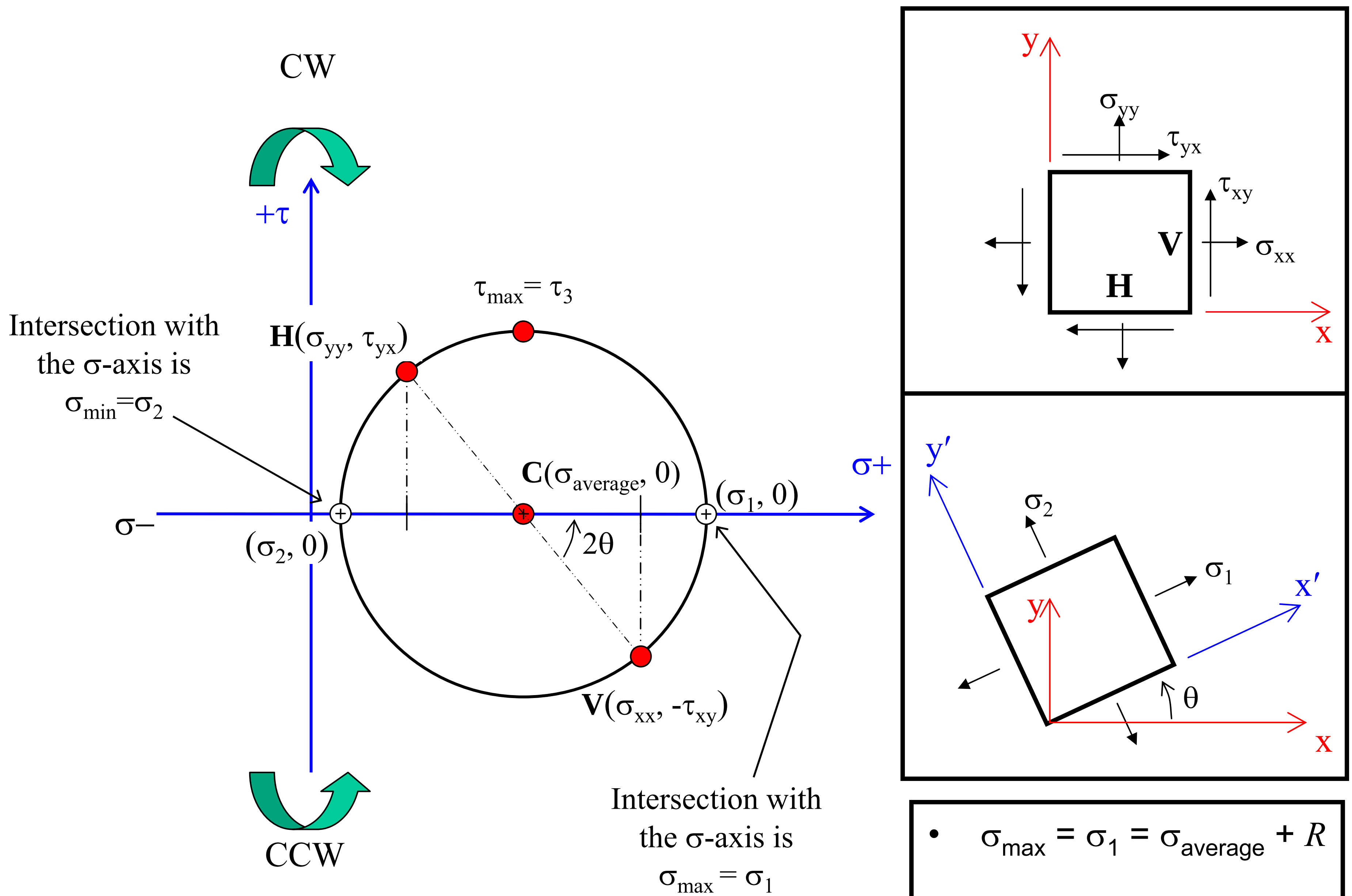
Steps in Construction of Mohr's Circle – cont'd

6. The angle between lines CV and Cσ_1 is labeled 2θ because the angles on Mohr's circle are double the actual angle between planes.

   To determine the direction of rotation (i.e., the sign) we first record the direction in which we move from point **V**(σ_{xx}, $-\tau_{xy}$) to point (σ_1, 0) on Mohr's circle.

   If the direction of rotation is CCW (i.e., towards the positive shear direction), then the sign of θ is positive. If the rotation is CW then the sign of θ is negative.

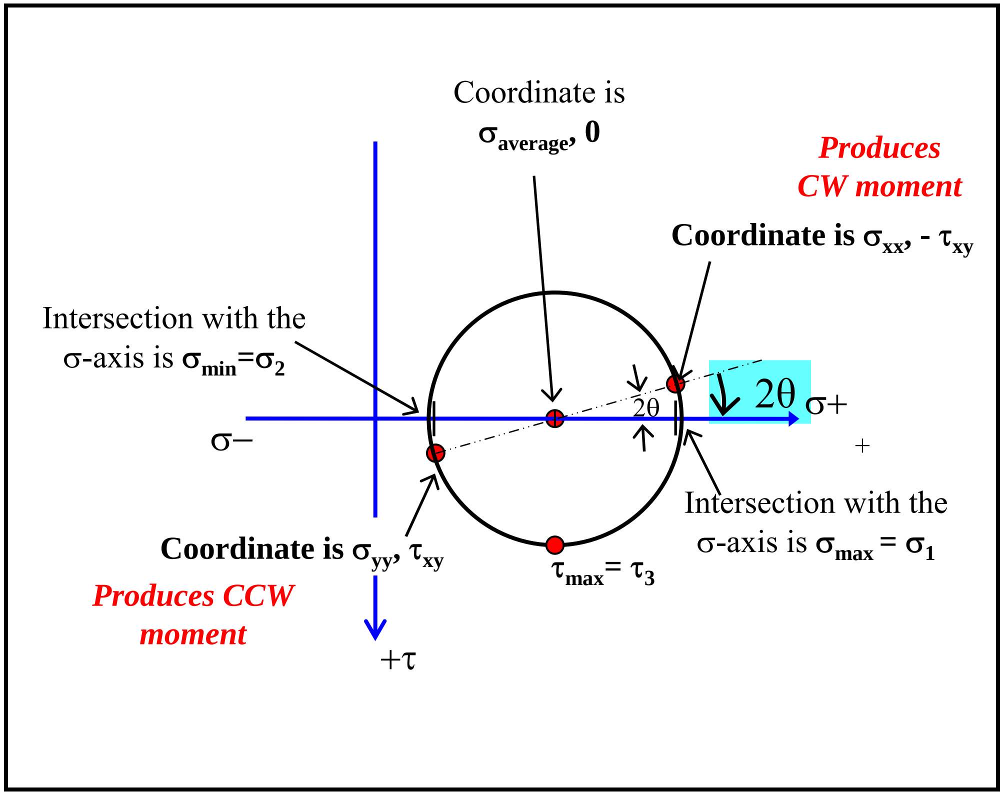
This is illustrated on the next viewgraph.

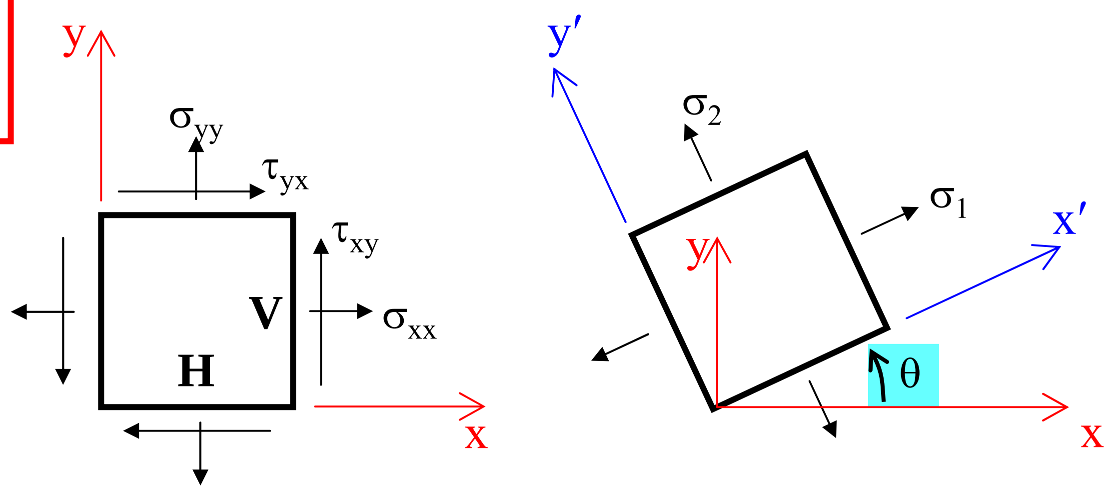


- $\sigma_{max} = \sigma_1 = \sigma_{average} + R$
- $\sigma_{min} = \sigma_2 = \sigma_{average} - R$

VERY IMPORTANT:

- Many engineering texts (and practicing engineers) construct Mohr's circle with shear stress pointing downwards as is illustrated to the right. In this case, the rotations between the principal stress axes and the state of stress on the volume element will be opposite of that on Mohr's circle.
- Be careful and know the system that your employer or the texts that you are referring to are using.





Example Problem

1. Consider a point in a solid that is subjected to the following state of stress:
   σ_{xx} = 90 MPa; σ_{yy} = -50 MPa; τ_{xy} = -60 MPa.

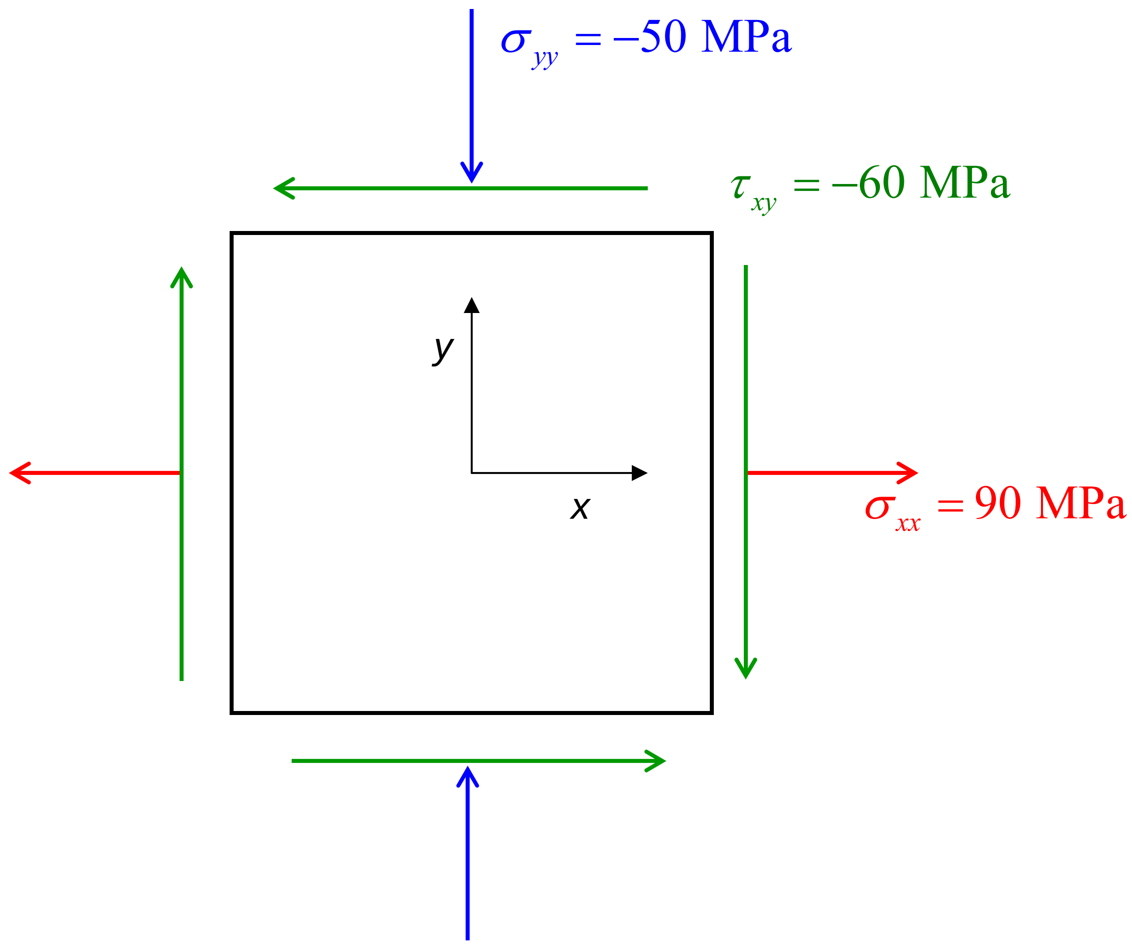
   a. Draw a free body diagram representing the stress state.

   b. Determine the principal stresses, the maximum in-plane shear stress acting on the point, and the orientation of the principal planes using Mohr's circle.

   c. Show the stresses no an appropriate diagram.

Example Problem - solution

a. Draw a free body diagram representing the stress state.



Example Problem – solution cont'd

b. Determine the principal stresses, the maximum in-plane shear stress acting on the point, and the orientation of the principal planes using Mohr's circle.

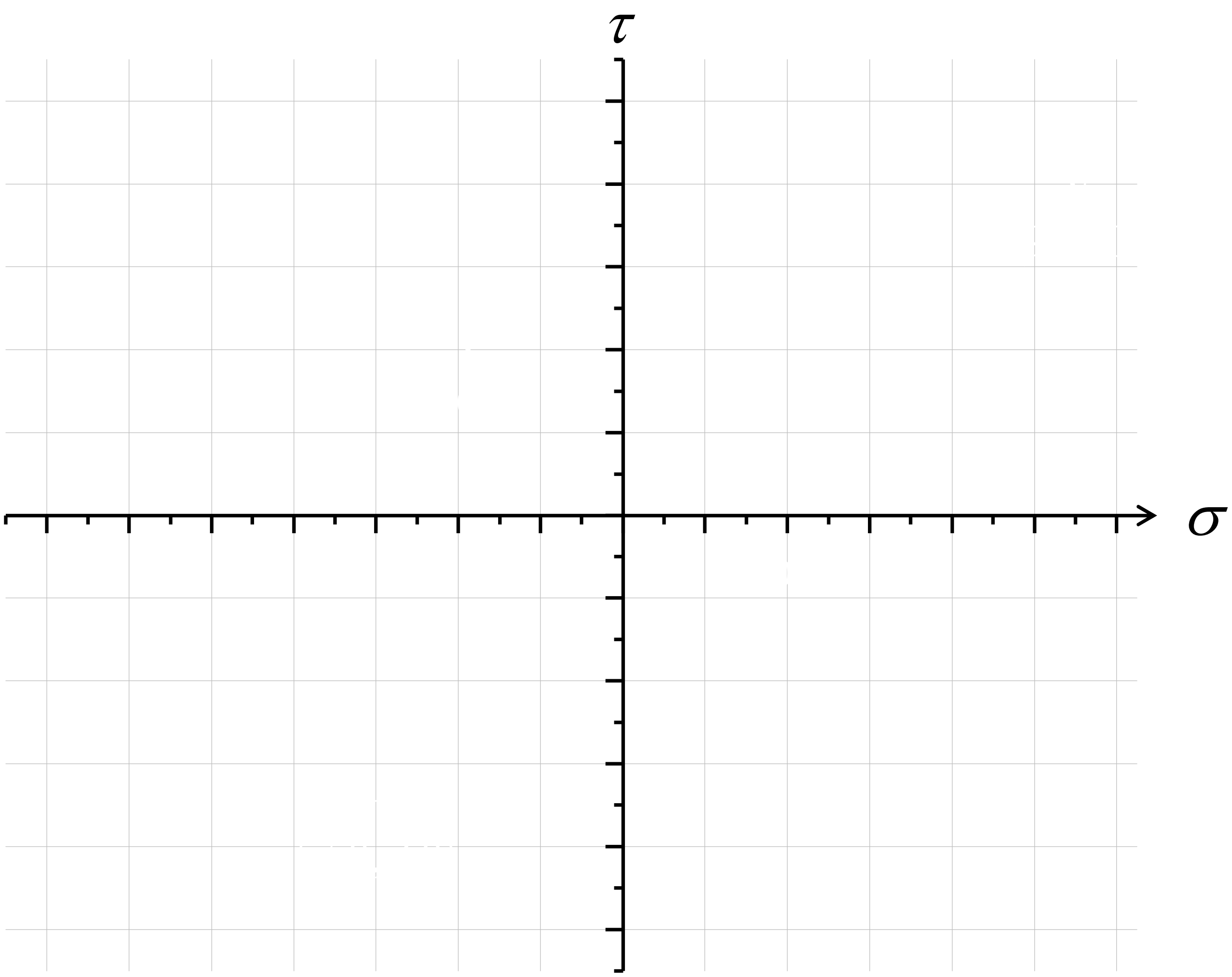
$$V = \left(\sigma_{xx}, -\tau_{xy}\right) = (90, 60)$$

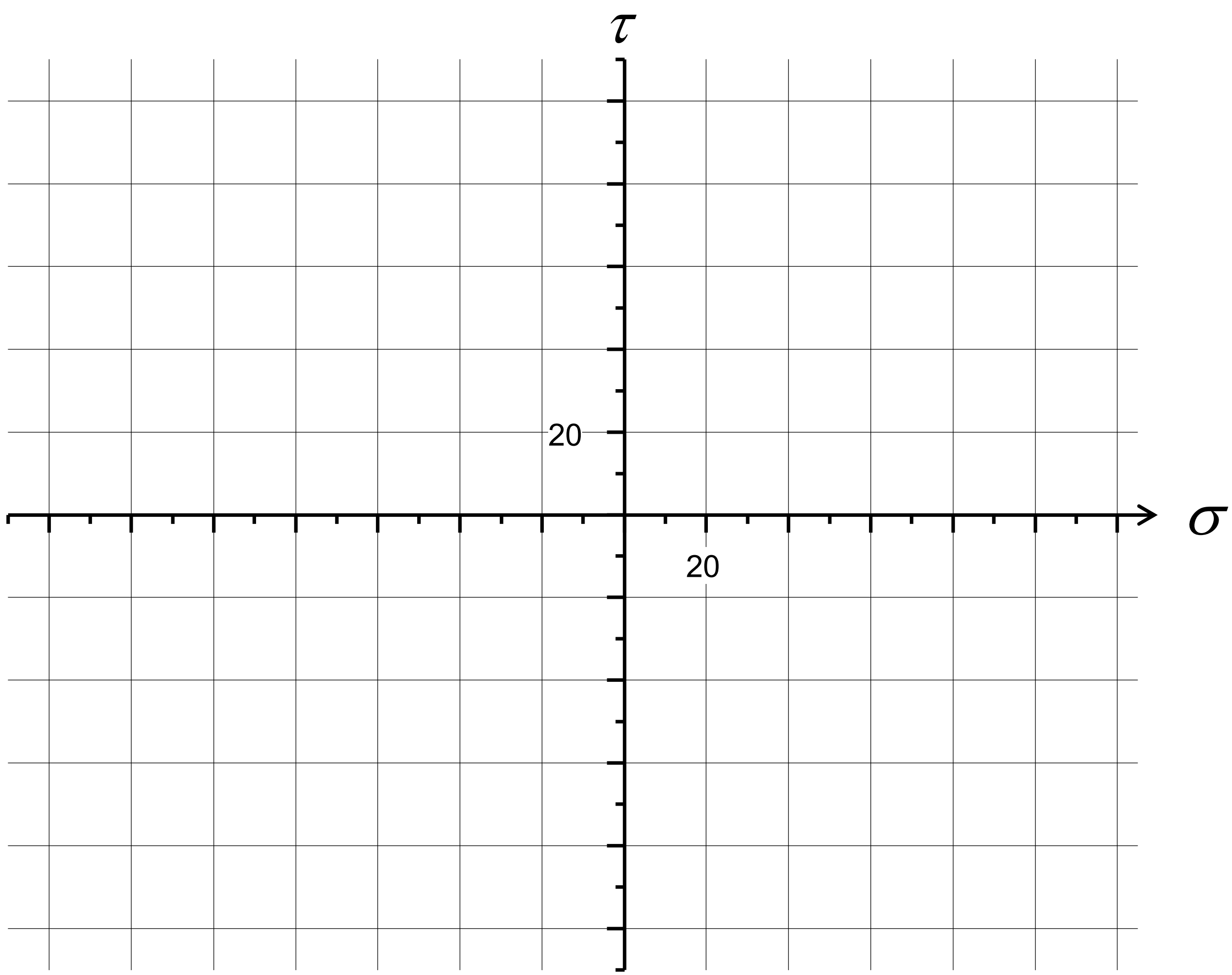
$$H = \left(\sigma_{yy}, +\tau_{xy}\right) = (-50, -60)$$

$$C = \left(\sigma_{average}, 0\right) = \left(\left(\frac{90 + (-50)}{2}\right), 0\right) = (20, 0)$$

$$R = \sqrt{\left(\frac{90 - (-50)}{2}\right)^2 + (-60)^2} = \sqrt{850} = 92.2 = \tau_{max}$$

solution cont'd

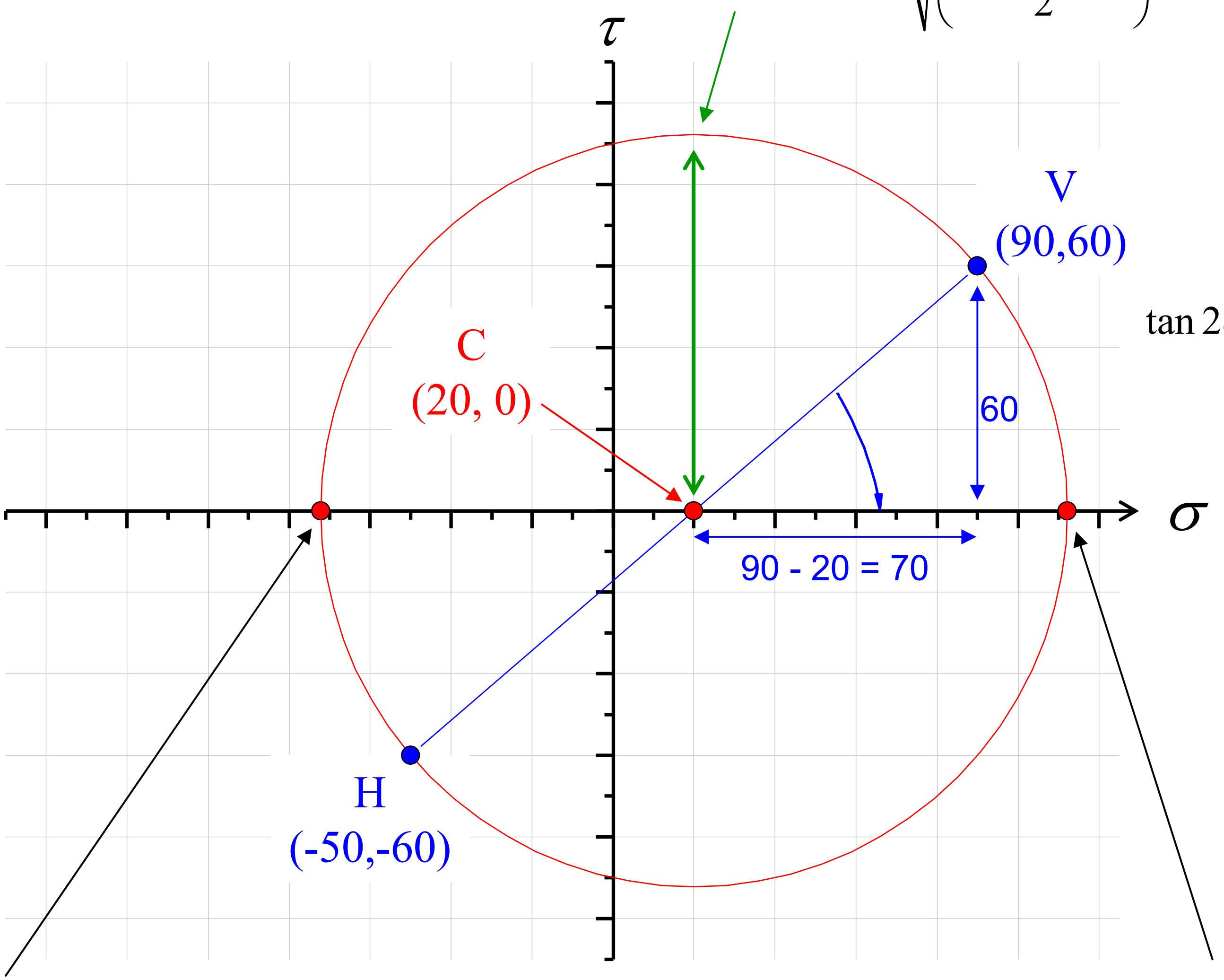




H*

solution cont'd

$$R = \tau_{\max} = \sqrt{\left(\frac{90-(-50)}{2}\right)^2 + (-60)^2} = \sqrt{850} = 92.2 \text{ MPa}$$



$$\tan 2\theta = \frac{\tau_{xy}}{\sigma_{xx} - \sigma_{avg}} = \frac{60}{70} = \frac{opp.}{adj.}$$

$$2\theta = -40.6°$$

$$\sigma_2 = \sigma_{avg} - R = -72.2 \text{ MPa}$$

$$\sigma_1 = \sigma_{avg} + R = 112.2 \text{ MPa}$$

B*

Example Problem - solution

c. Show the stresses on an appropriate diagram.