HOMEWORK
From Dieter
2-3, 2-4, 3-7

Module #3

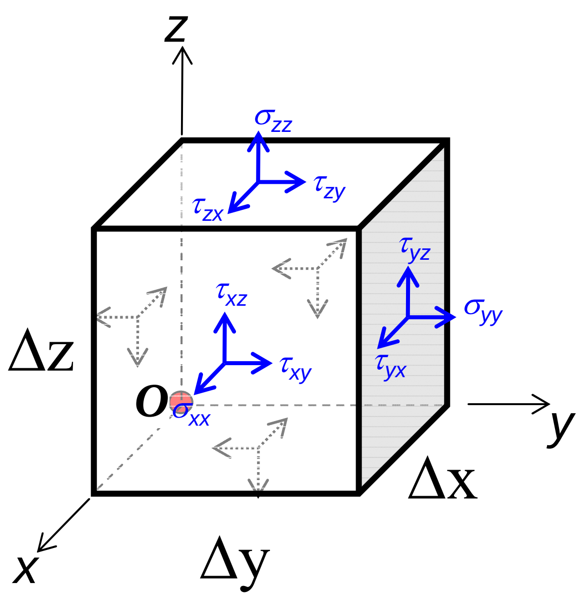
Transformation of stresses in 3-D

READING LIST

DIETER: Ch. 2, pp. 27-36

Ch. 3 in Roesler
Ch. 2 in McClintock and Argon
Ch. 7 in Edelglass

The Stress Tensor



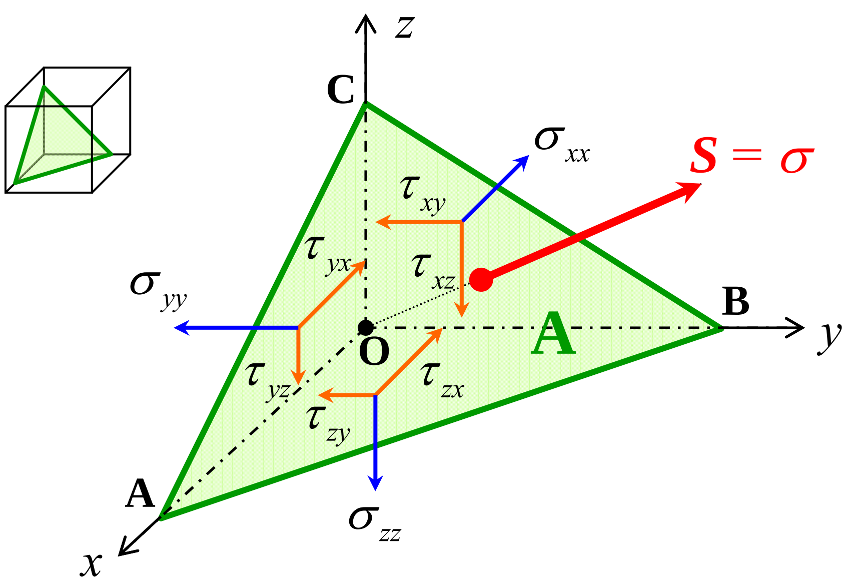
$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

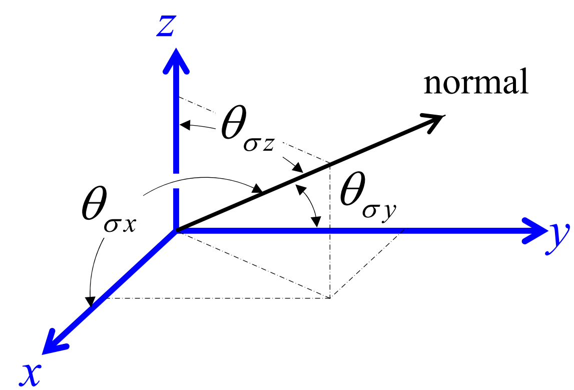
In three dimensions the state of stress is described by the stress tensor.

We can transform from one coordinate system to another in the same way that we did for two dimensions.

Method

- Lets resolve an arbitrary 3D state of stress onto an oblique plane *ABC* (area **A**).
- To make the problem easier, let ***S*** be parallel to the plane normal (meaning that it is a principal stress acting on a principal plane (i.e., the plane w/o shear).





DIRECTION COSINES

$$l = \cos \theta_{\sigma \cdot x}$$
$$m = \cos \theta_{\sigma \cdot y}$$
$$n = \cos \theta_{\sigma \cdot z}$$

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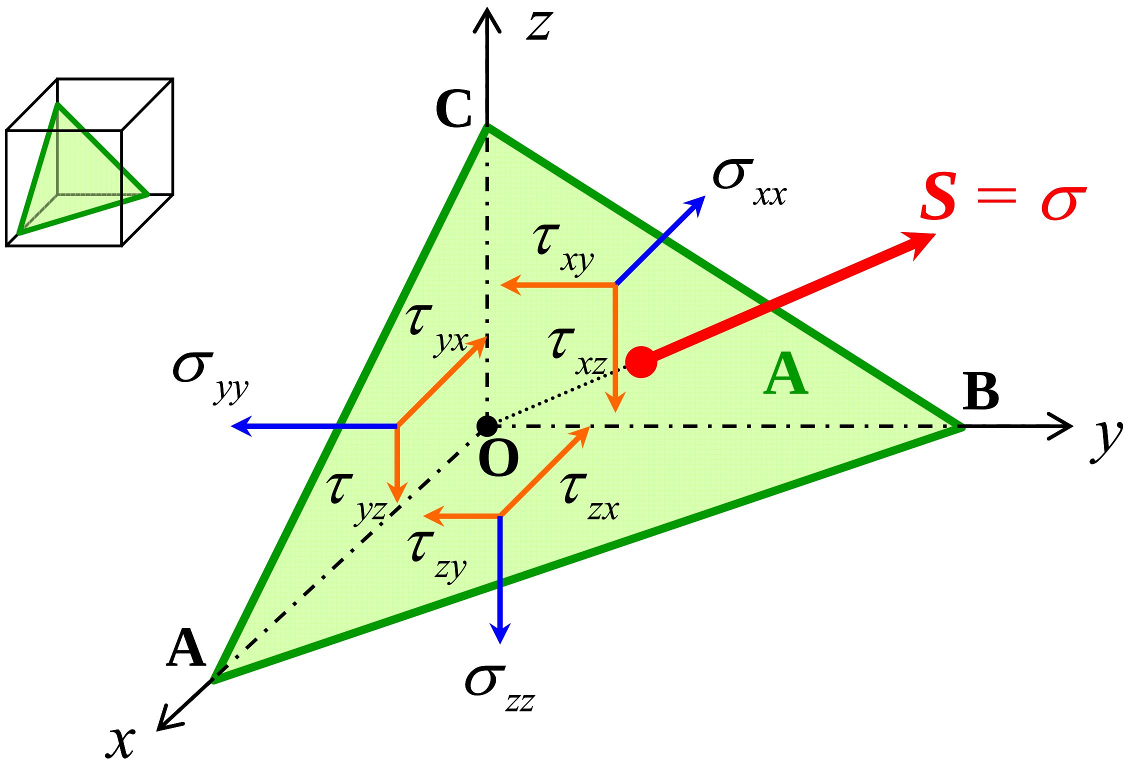
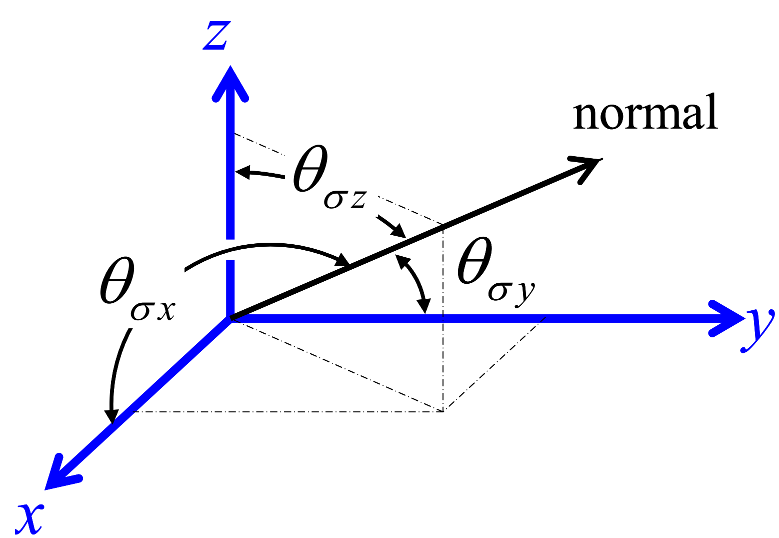


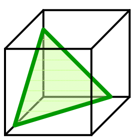


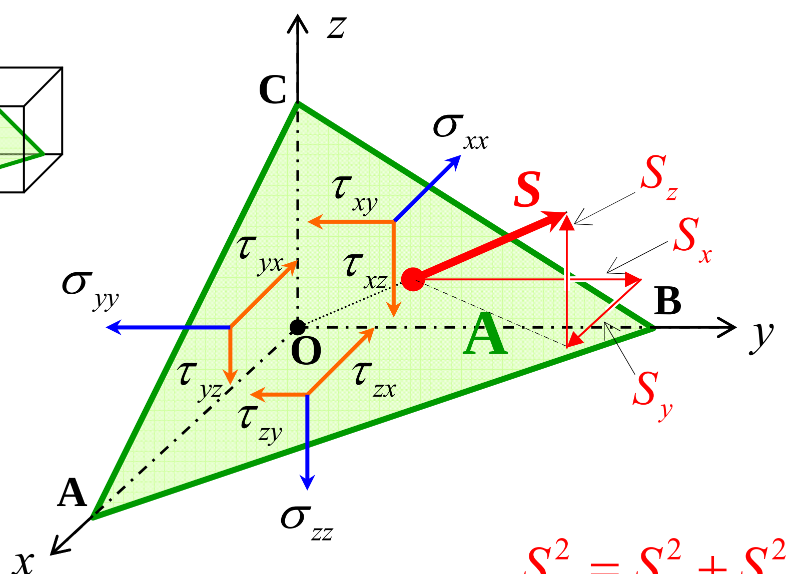
- The components of **S** parallel to the original x-, y-, and z-axes (i.e., $\mathbf{S}_x$, $\mathbf{S}_y$, $\mathbf{S}_z$) are:

  - $\mathbf{S}_x = \mathbf{S}l = \sigma l$
  - $\mathbf{S}_y = \mathbf{S}m = \sigma m$
  - $\mathbf{S}_z = \mathbf{S}n = \sigma n$

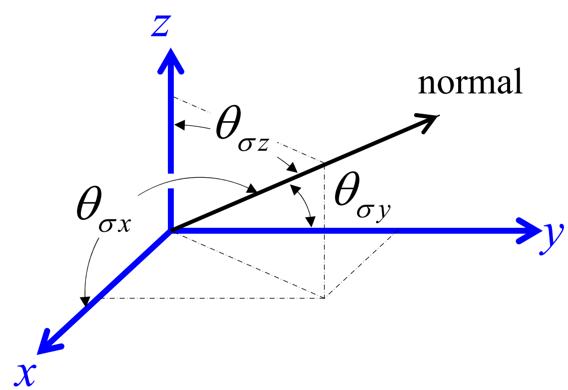
To balance force we need the areas that each stress components acts on

Area COB =
Area AOC =
Area AOB =





$$S^2 = S_x^2 + S_y^2 + S_z^2$$



Recall:

$l = \cos\,\theta_{\sigma \cdot x}$

$m = \cos\,\theta_{\sigma \cdot y}$

$n = \cos\,\theta_{\sigma \cdot z}$

All forces must balance to meet the conditions for static equilibrium (i.e., ∑F=0):

$$F_x = S_x A = SAl = \sigma_{xx} Al + \tau_{yx} Am + \tau_{zx} An$$
$$F_y = S_y A = SAm = \sigma_{yy} Am + \tau_{xy} Al + \tau_{zy} An$$
$$F_z = S_z A = SAn = \sigma_{zz} An + \tau_{xz} Al + \tau_{yz} Am$$

$$\downarrow$$

$$\sum F_x = (\sigma_{xx} - S)l + \tau_{yx} m + \tau_{zx} n = 0$$
$$\sum F_y = \tau_{xy} l + (\sigma_{yy} - S)m + \tau_{zy} n = 0$$
$$\sum F_z = \tau_{xz} l + \tau_{yz} m + (\sigma_{zz} - S)n = 0$$

$$\downarrow$$

$$\begin{bmatrix} (\sigma_{xx} - S) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_{yy} - S) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_{zz} - S) \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When written in matrix form

Method – cont'd

- The solution of the determinant of the matrix on the left yields a cubic equation in terms of ***S***.

$$S^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})S^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)S$$
$$-(\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{xz}^2 - \sigma_{zz}\tau_{xy}^2) = 0$$

or

$$S^3 - I_1 S^2 + I_2 S - I_3 = 0$$

- In this problem, $S = \sigma$.

- Thus, the three roots of this cubic equation represent the principal stresses, σ_1, σ_2, and σ_3.

Method – cont'd

- The directions in which the principal stresses act are determined by substituting σ_1, σ_2, and σ_3, each for S in:

$$(\sigma_{xx} - S)l + \tau_{yx}m + \tau_{zx}n = 0$$
$$\tau_{xy}l + (\sigma_{yy} - S)m + \tau_{zy}n = 0$$
$$\tau_{xz}l + \tau_{yz}m + (\sigma_{zz} - S)n = 0$$

Then the resulting equations must be solved simultaneously for l, m, and n (using the relationship $l^2+m^2+n^2 = 1$).

(a) Substitute σ_1 for S ; solve for l, m, and n;

(b) Substitute σ_2 for S ; solve for l, m, and n;

(c) Substitute σ_3 for S ; solve for l, m, and n.

Invariants of the Stress Tensor

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \tau_{yz} \\ \tau_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \tau_{xz} \\ \tau_{zx} & \sigma_{zz} \end{vmatrix} = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$

$$I_3 = \begin{vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{vmatrix} = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{xz}^2 - \sigma_{zz}\tau_{xy}^2$$

Whenever stresses are transformed from one coordinate system to another, these three quantities remain constant.

Example Problem #1

- Determine the principal normal stresses for the following state of stress:

$$\sigma_{xx} = 0, \quad \sigma_{yy} = 10, \quad \sigma_{zz} = -75,$$

$$\tau_{xy} = -50, \quad \tau_{yz} = \tau_{xz} = 0$$

or

$$\begin{bmatrix} 0 & -50 & 0 \\ -50 & 10 & 0 \\ 0 & 0 & -75 \end{bmatrix} \text{MPa}$$

Example Problem #1 – solution

- This problem can be solved by substituting the known state of stress into the cubic equation:

$$S^3 - I_1 S^2 + I_2 S - I_3 = 0$$

where $\boldsymbol{S} = \sigma$.

- This is detailed on the next page.

$$\begin{bmatrix} 0 & -50 & 0 \\ -50 & 10 & 0 \\ 0 & 0 & -75 \end{bmatrix} \text{MPa}$$

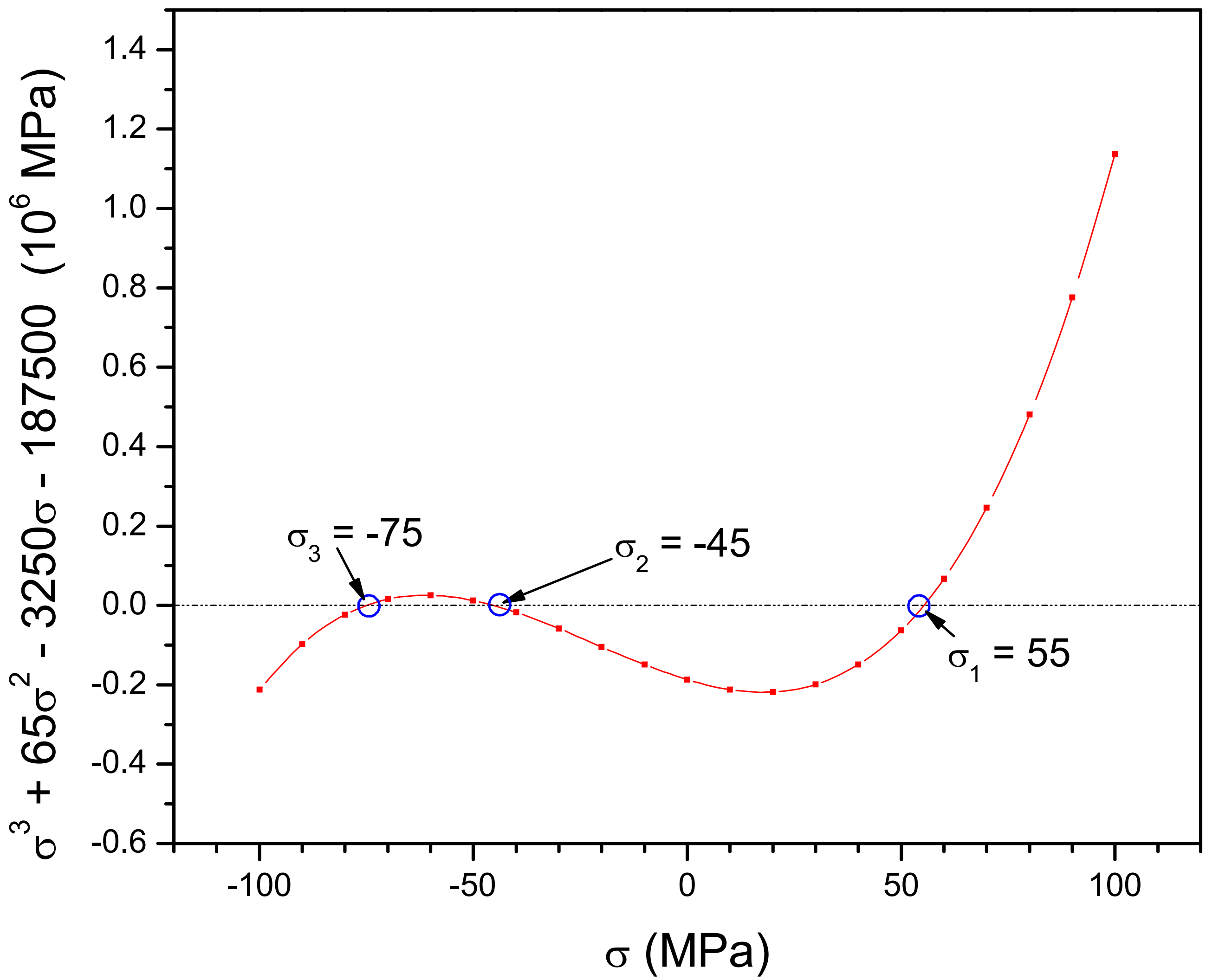
$$\sigma^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)\sigma$$
$$-(\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{xz}^2 - \sigma_{zz}\tau_{xy}^2) = 0$$

$$\sigma^3 - (0 + 10 + -75)\sigma^2 + [(0 \cdot 10) + (10 \cdot -75) + (0 \cdot -75) - (-50)^2 - (0)^2 - (0)^2]\sigma$$
$$-[(0 \cdot 10 \cdot -75) + 2(-50 \cdot 0 \cdot 0) - (0 \cdot 0^2) - (10 \cdot 0^2) - (-75 \cdot -50^2)] = 0$$

$$\sigma^3 - (\underset{\substack{\uparrow \\ I_1}}{-65})\sigma^2 + [\underset{\substack{\uparrow \\ I_2}}{-3250}]\sigma - [\underset{\substack{\uparrow \\ I_3}}{187500}] = 0$$

$$\sigma^3 + 65\sigma^2 - 3250\sigma - 187500 = 0$$

You can determine the principal stresses by plotting this equation <u>OR</u> you can solve it using more traditional means.



Example Problem #1 – solution

- This problem is easier than most because there are no shear stresses along the z-axis.

- It should have been obvious toyou that one of the principal stresses is σ = -75 MPa (since $\tau_{zx} = \tau_{xz} = 0$ and $\tau_{zy} = \tau_{yz} = 0$).

- Can you determine the directions in which the principal stresses act?

(I RECOMMEND THAT YOU TRY IT)

Resources on the Web

- There are many useful eigenvalue calculators on the world wide web. Here are a few:

- http://portal.cs.umass.edu/projects/mohr/

- http://www.engapplets.vt.edu/Mohr/java/nsfapplets/MohrCircles2-3D/Applets/applet.htm

H*

Example Problem #2

- Determine (a) the principal stresses, (b) maximum shear stress, and (c) the orientations of the principal planes for the state of stress provided below:

$$\begin{bmatrix} 80 & 20 & -50 \\ 20 & -40 & 30 \\ -50 & 30 & 60 \end{bmatrix} \text{MPa}$$

1

$$\sigma^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)\sigma$$
$$-(\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{xz}^2 - \sigma_{zz}\tau_{xy}^2) = 0$$

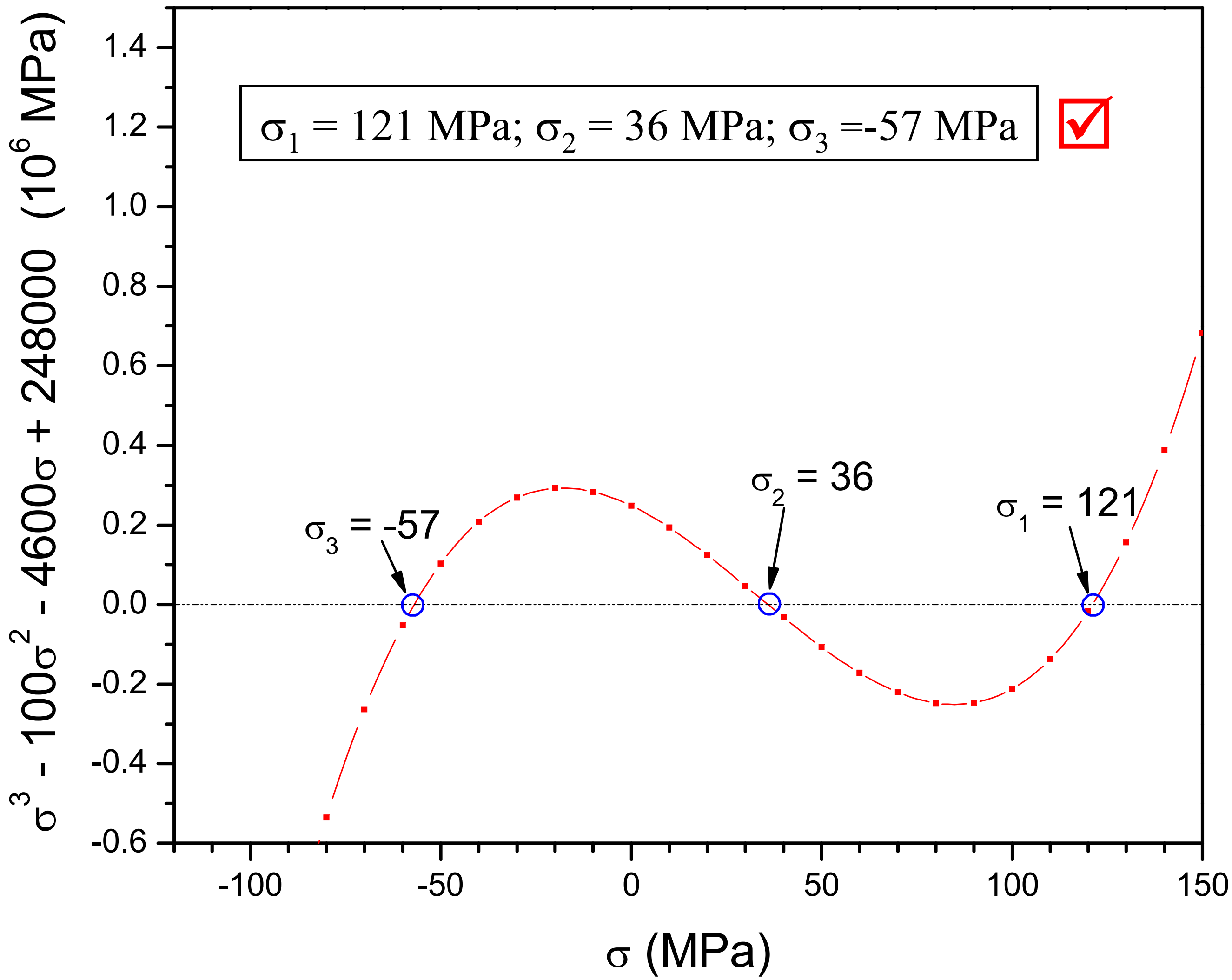
$$\sigma^3 - (80 - 40 + 60)\sigma^2 + [(80 \cdot -40) + (-40 \cdot 60) + (80 \cdot 60) - (20)^2 - (30)^2 - (50)^2]\sigma$$
$$-[(80 \cdot -40 \cdot 60) + 2(20 \cdot 30 \cdot 60) - (80 \cdot 30^2) - (-40 \cdot -50^2) - (60 \cdot 20^2)] = 0$$

$$\sigma^3 - (100)\sigma^2 + [-4600]\sigma - [-248000] = 0$$

$$\sigma^3 - 100\sigma^2 - 4600\sigma + 248000 = 0$$

$$\sigma^3 - 100\sigma^2 - 4600\sigma + 248000 = 0$$

1



$$\begin{bmatrix} 80 & 20 & -50 \\ 20 & -40 & 30 \\ -50 & 30 & 60 \end{bmatrix} \text{MPa}$$

$$(\sigma_{xx} - \sigma)l + \tau_{yx}m + \tau_{zx}n = 0$$
$$\tau_{xy}l + (\sigma_{yy} - \sigma)m + \tau_{zy}n = 0$$
$$\tau_{xz}l + \tau_{yz}m + (\sigma_{zz} - \sigma)n = 0$$

$$\downarrow$$

$$(80 - \sigma)l + 20m - 50n = 0$$
$$20l + (-40 - \sigma)m + 30n = 0$$
$$-50l + 30m + (60 - \sigma)n = 0$$

substitute σ_1, σ_2, and σ_3 in place of σ and solve simultaneous equations

$\sigma_1 = 121$

$(80 - \sigma)l + 20m - 50n = 0$		$-41l + 20m - 50n = 0$	[1]
$20l + (-40 - \sigma)m + 30n = 0$	$\rightarrow$	$20l - 161m + 30n = 0$	[2]
$-50l + 30m + (60 - \sigma)n = 0$		$-50l + 30m - 61n = 0$	[3]

2

(3 × [1]) + (5 × [2]) yields:

$$-123l + 60m - 150n = 0$$
$$100l - 805m + 150n = 0$$
$$\overline{-23l - 745m + 0n = 0;} \quad \therefore \quad m = -0.0309l$$

Sample Problem #2
cont'd

substitute $\boxed{\sigma_1}$, σ_2, and σ_3 in place of σ and solve simultaneous equations

$\sigma_1 = 121$

$$\begin{aligned} (80-\sigma)l + 20m - 50n &= 0 & & -41l + 20m - 50n = 0 \quad [1] \\ 20l + (-40-\sigma)m + 30n &= 0 & \rightarrow & \ 20l - 161m + 30n = 0 \quad [2] \\ -50l + 30m + (60-\sigma)n &= 0 & & -50l + 30m - 61n = 0 \quad [3] \end{aligned}$$

3

(3 × [1]) + (-2 × [3]) yields:

$$\begin{aligned} -123l + 60m - 150n &= 0 \\ 100l - 60m + 122n &= 0 \\ \hline -23l + 0m - 28n &= 0; \end{aligned} \qquad \therefore \ n = -0.821l$$

Sample Problem #2 cont'd

$$l^2 + m^2 + n^2 = 1$$

substitute expresions for m & n

$$l^2 + \left[-0.031l\right]^2 + \left[-0.821l\right]^2 = 1.673l^2 = 1$$

4

$$l = \sqrt{\frac{1}{1.673}} = \boxed{0.773}$$

$$m = -0.031l = \boxed{-0.024}$$

$$n = -0.821l = \boxed{-0.635}$$

Orientations of principal planes associated with σ_1 ☑

Sample Problem #2
cont'd

substitute σ_1, σ_2, and σ_3 in place of σ and solve simultaneous equations

$\sigma_2 = 36$

$$(80-\sigma)l + 20m - 50n = 0 \qquad\qquad 44l + 20m - 50n = 0 \quad [4]$$

$$20l + (-40-\sigma)m + 30n = 0 \quad \rightarrow \quad 20l - 76m + 30n = 0 \quad [5]$$

$$-50l + 30m + (60-\sigma)n = 0 \qquad\qquad -50l + 30m + 24n = 0 \quad [6]$$

5

(3 × [4]) + (5 × [5]) yields:

$$132l + 60m - 150n = 0$$

$$100l - 380m + 150n = 0$$

$$232l - 320m - 0n = 0; \quad \therefore \quad m = 0.725l$$

Sample Problem #2
cont'd

substitute σ_1, σ_2, and σ_3 in place of σ and solve simultaneous equations

$\sigma_2 = 36$

$$\begin{aligned} (80-\sigma)l + 20m - 50n &= 0 \\ 20l + (-40-\sigma)m + 30n &= 0 \\ -50l + 30m + (60-\sigma)n &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} 44l + 20m - 50n &= 0 \quad [4] \\ 20l - 76m + 30n &= 0 \quad [5] \\ -50l + 30m + 24n &= 0 \quad [6] \end{aligned}$$

6

(3 × [4]) + (-2 × [6]) yields:

$$\begin{aligned} 132l + 60m - 150n &= 0 \\ 100l - 60m - 48n &= 0 \\ \hline 232l + 0m - 198n &= 0; \end{aligned} \quad \therefore \quad n = 1.172l$$

Sample Problem #2
cont'd

$$l^2 + m^2 + n^2 = 1$$

substitute expresions for m & n

$$l^2 + [0.725l]^2 + [1.172l]^2 = 2.899l^2$$

7

$$l = \sqrt{\frac{1}{2.899}} = \boxed{0.587}$$

$$m = 0.725l = \boxed{0.426}$$

$$n = 1.172l = \boxed{0.688}$$

Orientations of principal planes associated with σ_2 ☑

Sample Problem #2
cont'd

substitute σ_1, σ_2, and σ_3 in place of σ and solve simultaneous equations

$\sigma_3 = -57$

$$
\begin{aligned}
(80-\sigma)l + 20m - 50n &= 0 \\
20l + (-40-\sigma)m + 30n &= 0 \\
-50l + 30m + (60-\sigma)n &= 0
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
137l + 20m - 50n &= 0 \quad [7] \\
20l + 17m + 30n &= 0 \quad [8] \\
-50l + 30m + 117n &= 0 \quad [9]
\end{aligned}
$$

8

(3 × [7]) + (5 × [8]) yields:

$$
\begin{aligned}
411l + 60m - 150n &= 0 \\
100l + 85m + 150n &= 0 \\
\hline
511l + 145m - 0n &= 0;
\end{aligned}
\quad \therefore \quad m = -3.524l
$$

Sample Problem #2
cont'd

substitute σ_1, σ_2, and σ_3 in place of σ and solve simultaneous equations

$\sigma_3 = -57$

$$\begin{aligned} (80-\sigma)l + 20m - 50n &= 0 \\ 20l + (-40-\sigma)m + 30n &= 0 \\ -50l + 30m + (60-\sigma)n &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} 137l + 20m - 50n &= 0 \quad [7] \\ 20l + 17m + 30n &= 0 \quad [8] \\ -50l + 30m + 117n &= 0 \quad [9] \end{aligned}$$

9

(3 × [7]) + (-2 × [6]) yields:

$$\begin{aligned} 411l + 60m - 150n &= 0 \\ 100l - 60m - 234n &= 0 \\ \hline 511l + 0m - 384n &= 0; \end{aligned} \quad \therefore \quad n = 1.331l$$

Sample Problem #2
cont'd

$$l^2 + m^2 + n^2 = 1$$

substitute expresions for m & n

$$l^2 + [-3.524l]^2 + [1.331l]^2 = 15.189l^2$$

10

$$l = \sqrt{\frac{1}{15.189}} = \boxed{0.257}$$

$$m = -3.524l = \boxed{-0.904}$$

$$n = -1.331l = \boxed{0.342}$$

Orientations of principal planes associated with σ_3 ☑

10

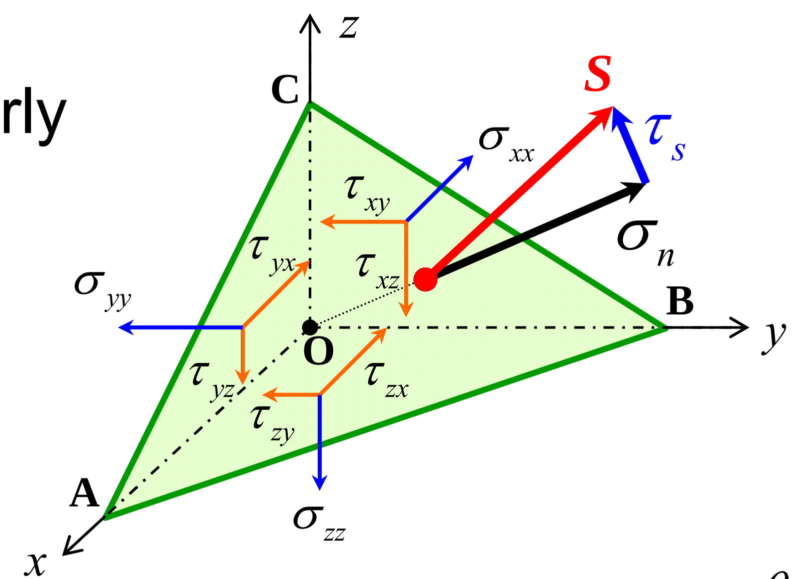
$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{121 - (-57)}{2} = \boxed{89 \text{ MPa}}$$ ☑

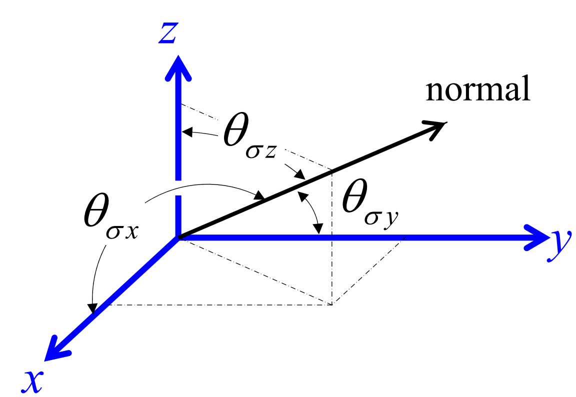
5 minute break

General Method for Triaxial States of Stress

[p. 29-30 in Dieter]

- In previous example/method, we assumed that the stress on the inclined plane was a principal stress.
- What if the stress on the new plane is not a principal stress?
- The math is nearly the same.





$$S^2 = \sigma_n^2 + \tau_s^2 = S_x^2 + S_y^2 + S_z^2$$

Triaxial Stress States – cont'd

From summation of forces parallel to the x, y, z axes, we find the components S_x, S_y, S_z:

$$S_x = \sigma_{xx} l + \tau_{yx} m + \tau_{zx} n$$

$$S_y = \tau_{xy} l + \sigma_{yy} m + \tau_{zy} n$$

$$S_z = \tau_{xz} l + \tau_{yz} m + \sigma_{zz} n$$

The normal stress on the oblique plane equals the sum of the components S_x, S_y, S_z parallel to the plane normal.

$$\sigma_n = S_x l + S_y m + S_z n$$

$$= \sigma_{xx} l^2 + \sigma_{yy} m^2 + \sigma_{zz} n^2 + 2lm\tau_{xy} + 2mn\tau_{yz} + 2nl\tau_{zx}$$

Triaxial Stress States – cont'd

From the expression $S^2 = \sigma_n^2 + \tau_s^2$, the shear stress can be obtained.

When written in terms of principal axes, it becomes:

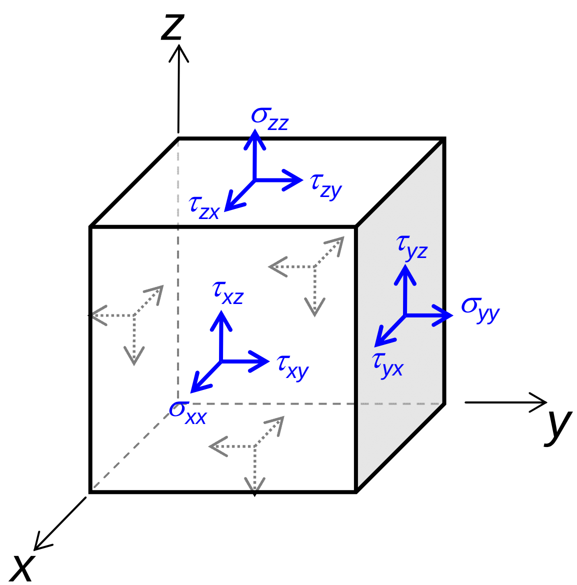
$$\tau_s^2 = (\sigma_1 - \sigma_2)^2 l^2 m^2 + (\sigma_1 - \sigma_3)^2 l^2 n^2 + (\sigma_2 - \sigma_3)^2 m^2 n^2$$

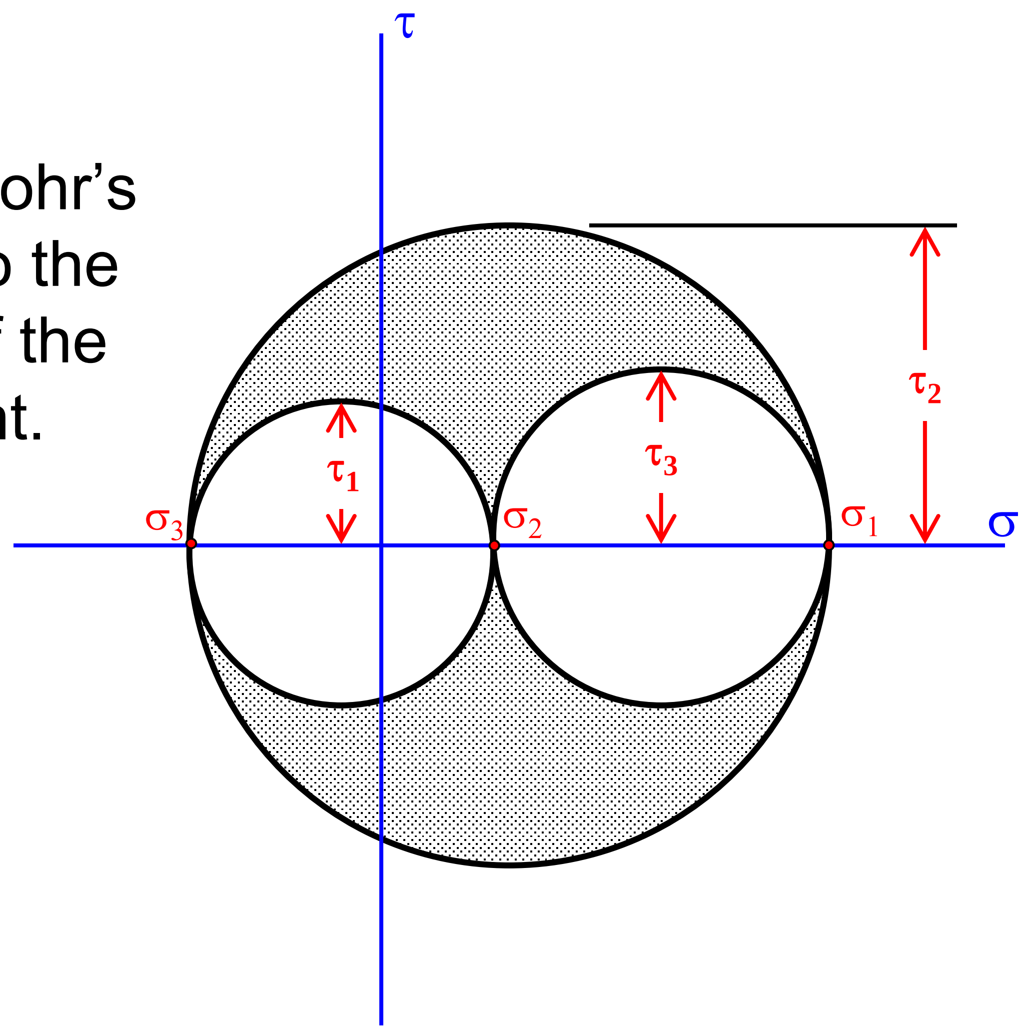
The maximum shear stress occurs when:

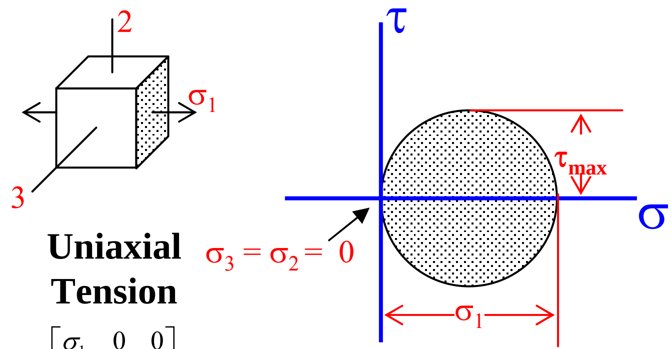
$$\boxed{\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{\max} - \sigma_{\min}}{2}}$$

Mohr's Circle in 3-D

- We can use a 3-D Mohr's circle to visualize the state of stress and to determine principal stresses.
- Essentially three 2-D Mohr's circles corresponding to the *x-y, x-z, and y-z* faces of the elemental cubic element.

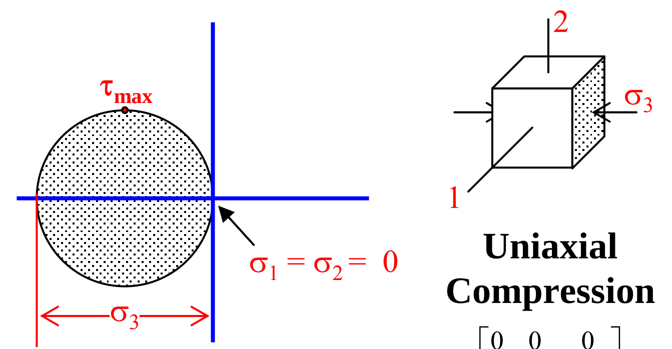






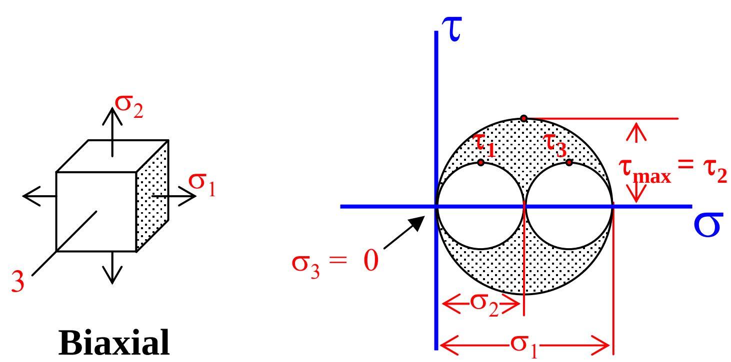
Uniaxial Tension

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



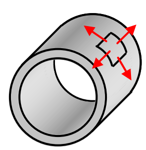
Uniaxial Compression

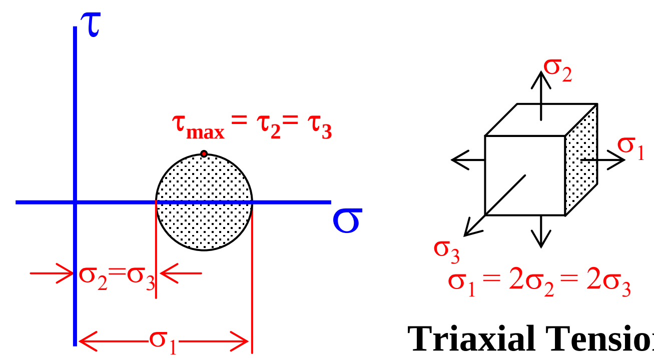
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sigma_3 \end{bmatrix}$$



Biaxial Tension

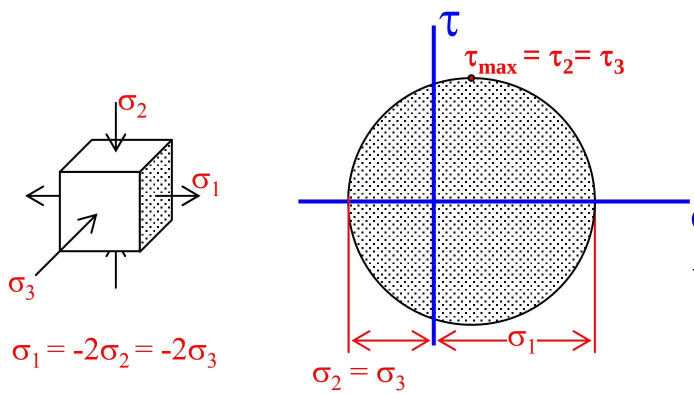
$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$





Triaxial Tension (unequal)

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$



Uniaxial Tension plus Biaxial Compression

Adapted from G.E. Dieter, <u>Mechanical Metallurgy</u>, 3rd ed., McGraw-Hill (1986) p. 37