HOMEWORK
From Dieter
2-7

Module #4

Fundamentals of strain
The strain deviator
Mohr's circle for strain

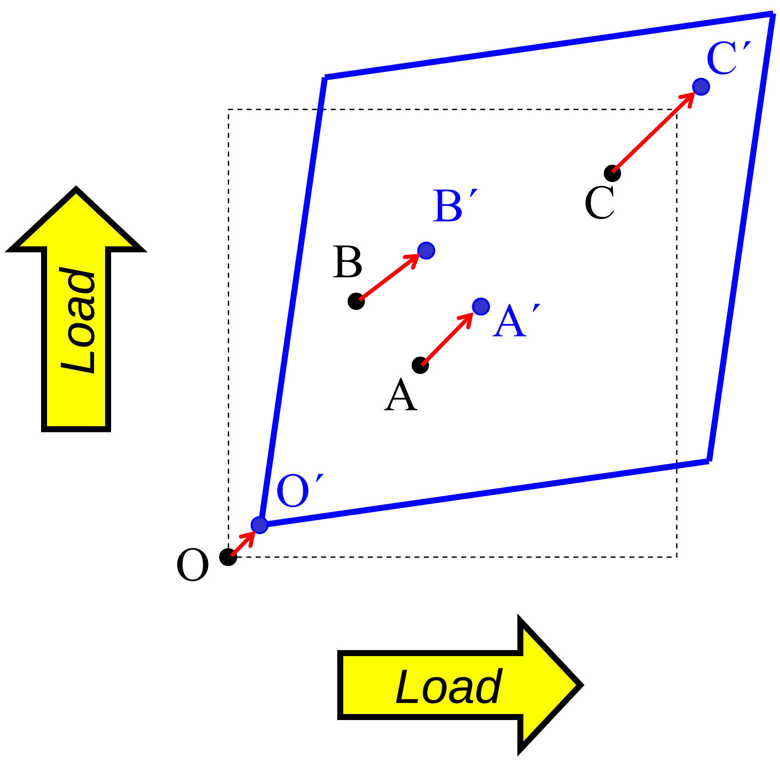
READING LIST

DIETER: Ch. 2, Pages 38-46

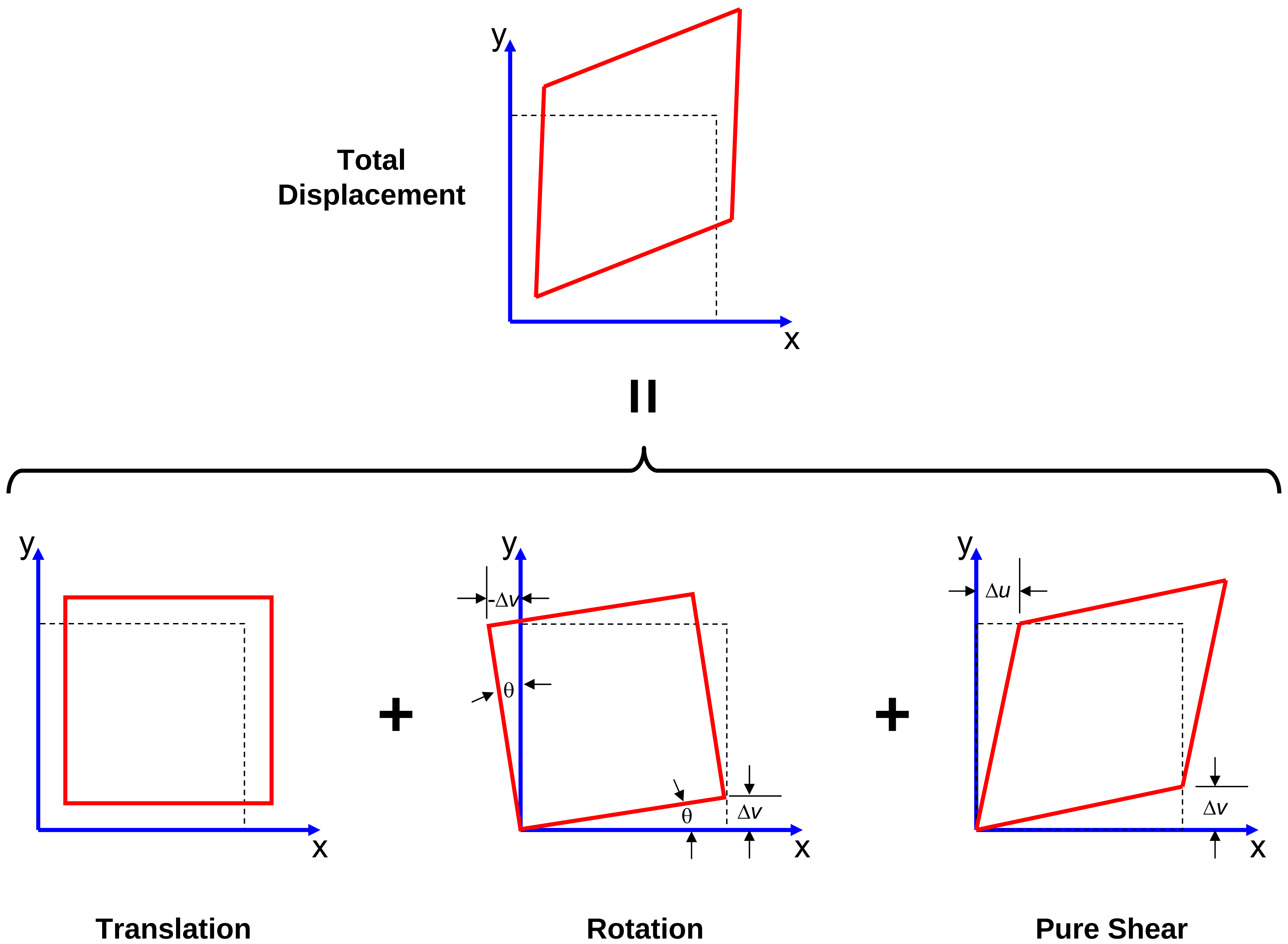
Pages 11-12 in Hosford
Ch. 6 in Nye

Strain

- When a solid is subjected to a load, parts of the solid are displaced from their original positions.
- Think of it like this; the atoms making up the solid are displaced from their original positions.



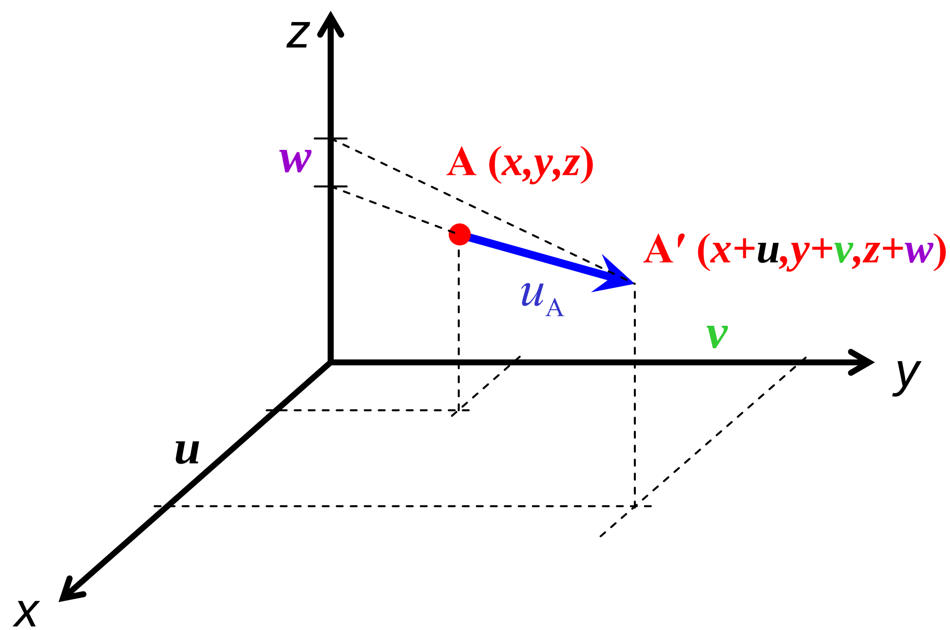
- This displacement of points or particles under an applied stress is termed strain.



Total Displacement
y
x
=
y
x
Translation
+
y
x
-Δv
θ
θ
Δv
Rotation
+
y
x
Δu
Δv
Pure Shear

Where to begin

- Consider a point **A** in a solid located at position ***x,y,z***.



- Apply force to the body and point **A (*x,y,z*)** is displaced to **A′ (*x*+*u*,*y*+*v*,*z*+*w*)**.

Displacement of points

- Displacement vector:

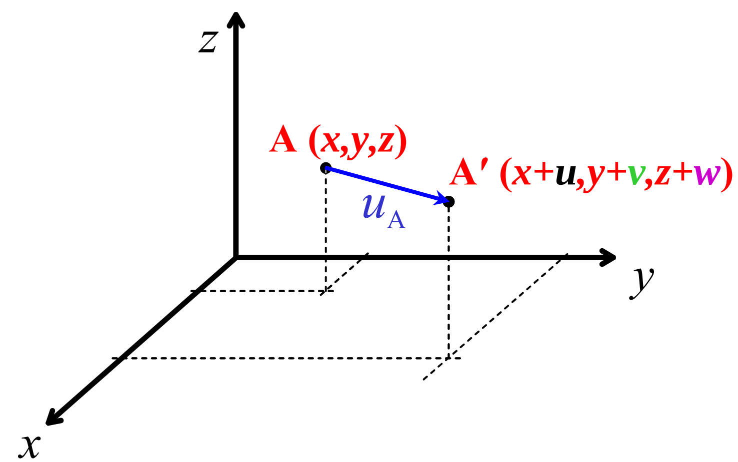
  u_A = f(u, v, w)

  where u, v, and w are units of translation along the x, y, z axes.

- Solids are composed of many particles.

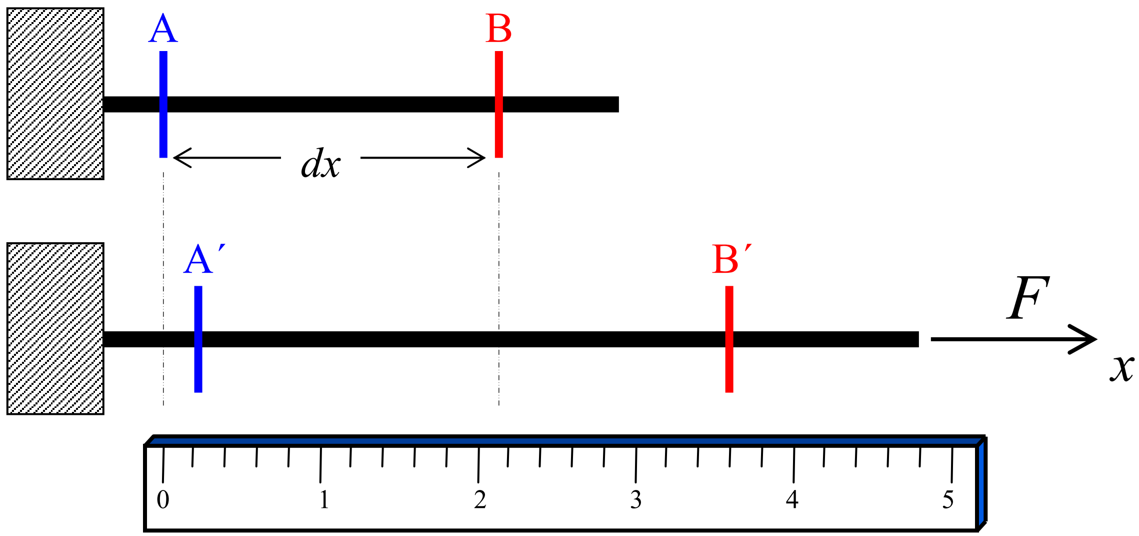
- If u_A is constant for all particles, no deformation occurs (only translation).

- If u_A varies from particle to particle, *i.e.*, u_i = f(x_i), the solid deforms.



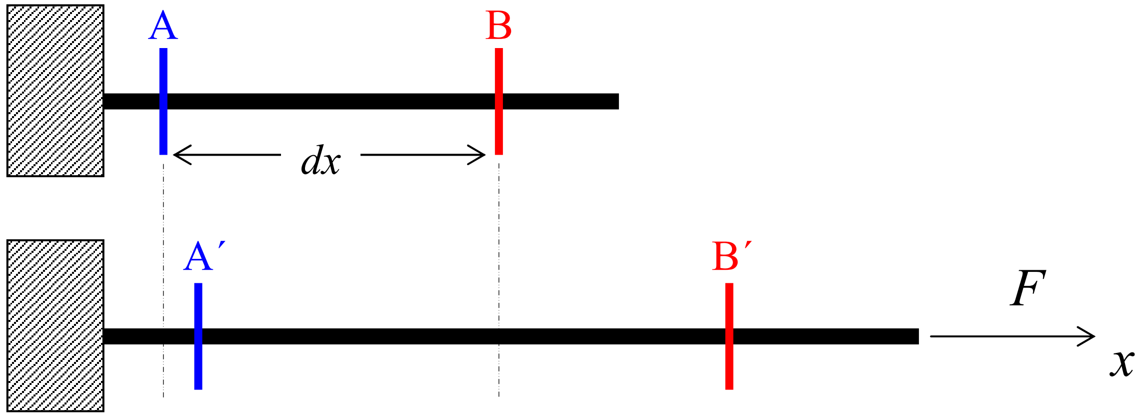
Displacement = Translation + Rotation + Shear

1-D Linear Strain



- Points A and B are displaced from their original positions
- The amount of displacement is a function of x. Point B moves farther than Point A.
  - Let distance A $\rightarrow$ A′ = u.
  - Thus, distance B $\rightarrow$ B′ = $u+(\Delta u/\Delta x)dx$ = $u+(\partial u/\partial x)dx$.

1-D Linear Strain – cont'd.



- Strain is defined by the following relationship:

$$e_{xx} = \frac{\Delta L}{L} = \frac{A'B' - AB}{AB} = \frac{dx + \frac{\partial u}{\partial x}dx - dx}{dx} = \frac{\partial u}{\partial x} \quad \left(= \frac{\Delta u}{\Delta x} \right)$$

- Integrating yields the displacement.

$$u = u_o + e_{xx}x$$

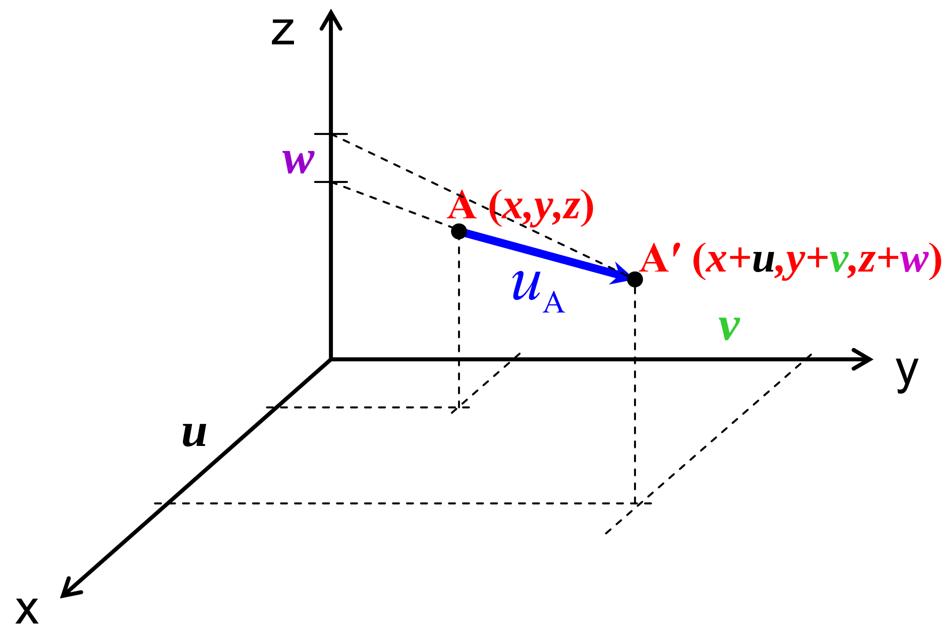
- $u_o \approx$ rigid body translation, which we can subtract, yielding:

$$u = e_{xx}x$$

Generalization to 3-D

- Displacement is related to the initial coordinates of the point.

$$\left.\begin{aligned} u &= e_{xx}x + e_{xy}y + e_{xz}z \\ v &= e_{yx}x + e_{yy}y + e_{yz}z \\ w &= e_{zx}x + e_{zy}y + e_{zz}z \end{aligned}\right\} u_i = e_{ij}x_j$$



- Normal / Linear Strains:

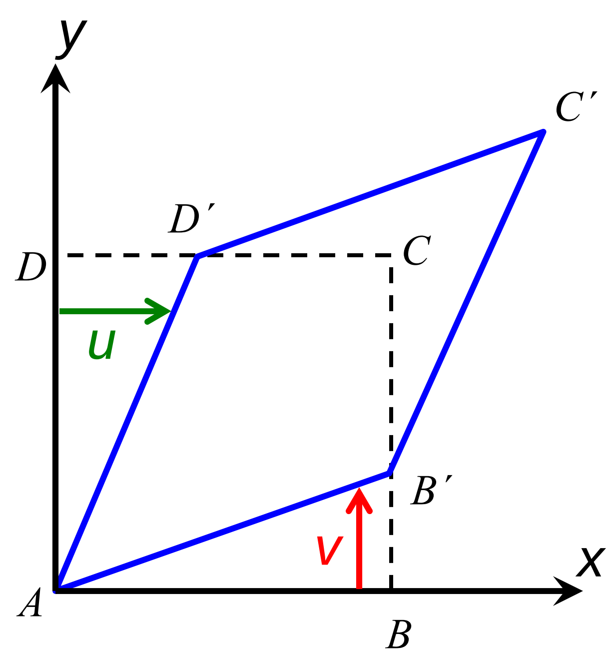
$$e_{xx} = \frac{\partial u}{\partial x},\; e_{yy} = \frac{\partial v}{\partial y},\; e_{zz} = \frac{\partial w}{\partial z}$$

$$\left[e_{xx} = \frac{\Delta u}{\Delta x},\; e_{yy} = \frac{\Delta v}{\Delta y},\; e_{zz} = \frac{\Delta w}{\Delta z} \right]$$

If we orient the system such that the load is applied parallel to the x-axis. The variables *u, v,* and *w* are displacements parallel to the *x*, *y*, and *z* axes.

Shear Strains in 2-D and 3-D

- Consider a square or cubic element that is distorted by shear.



Incremental displacement in x-direction = u.

Incremental displacement in y-direction = v.

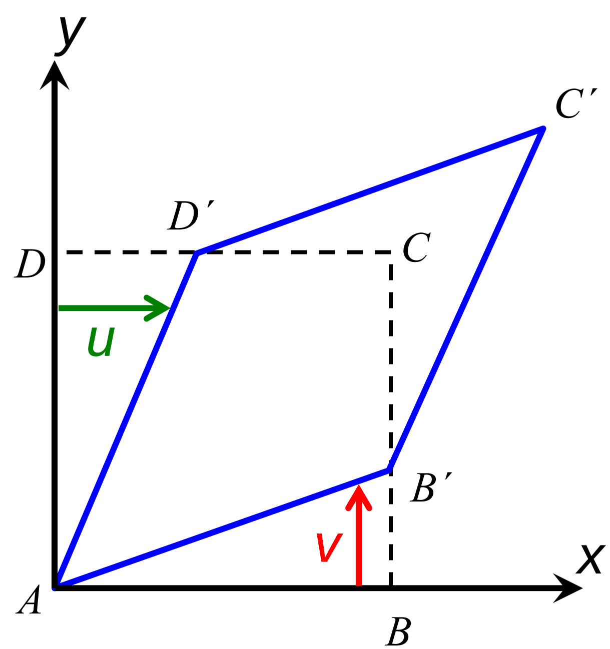
Incremental displacement in z-direction = w.

2D

- *Displacement of AD increases with distance along the y-axis resulting in an angular distortion of y-axis.*

$$e_{xy} = \frac{\delta}{h} = \frac{DD'}{DA} = \frac{\partial u}{\partial y} \quad \left[\text{or } \frac{\Delta u}{\Delta y} \right]$$

shear distortion of the y-axis in the x-direction



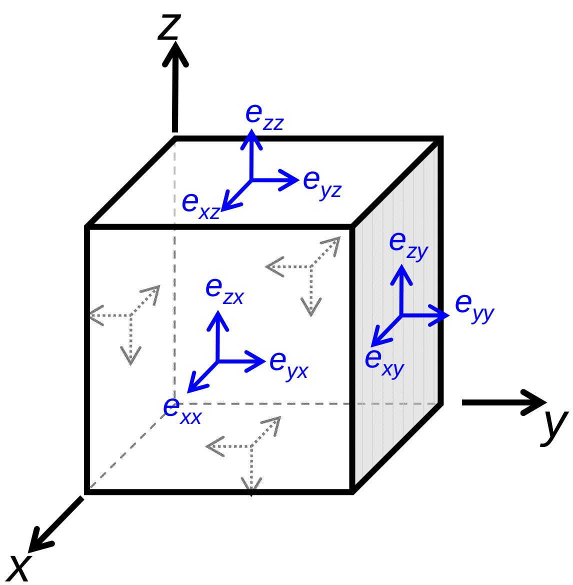
- *An analogous event occurs along the x-axis.*

$$e_{yx} = \frac{\delta}{h} = \frac{BB'}{AB} = \frac{\partial v}{\partial x} \quad \left[\text{or } \frac{\Delta v}{\Delta x} \right]$$

shear distortion of the x-axis in the y-direction

Strain in 3-D

- The *displacement strain* is defined by nine strain components:
  - e_{xx}, e_{xy}, e_{xz}, e_{yy}, e_{yx}, e_{yz}, e_{zz}, e_{zx}, e_{zy}
  - The strains on the negative faces are equal to satisfy the requirements for equilibrium.



- **Notation** is similar to stress; subscripts reversed:
  - e_{ij}: i = direction of displacement
    j = plane on which strain acts

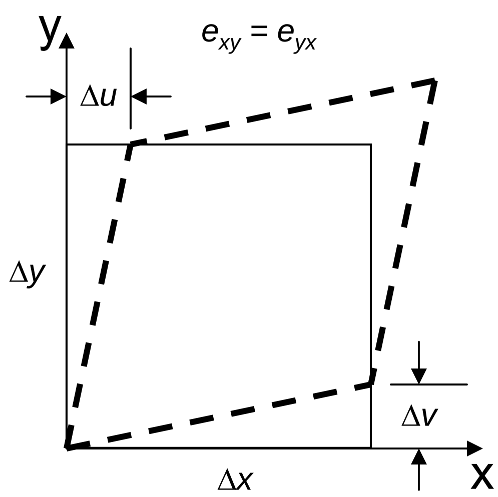
- **Convention**
  - (+)ive when both *i* & *j* are (+)ive
  - (+)ive when both *i* & *j* are (-)ive
  - (-)ive when both *i* & *j* are opposite

- Tension: e_{ij} = positive
- Compression: e_{ij} = negative

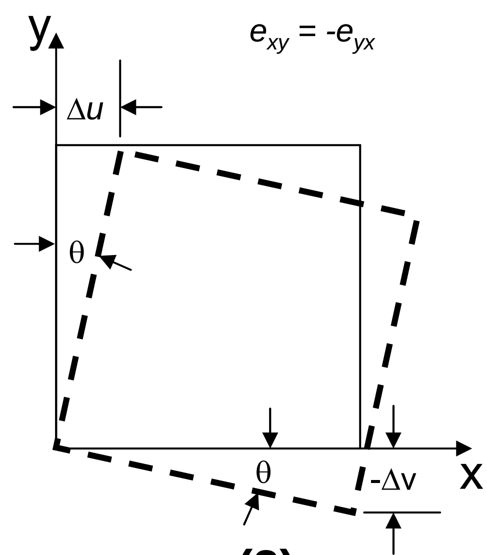
3D Displacement Strain Matrix

$$e_{ij} = \begin{vmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{vmatrix} = \begin{vmatrix} \frac{\Delta u}{\Delta x} & \frac{\Delta u}{\Delta y} & \frac{\Delta u}{\Delta z} \\ \frac{\partial v}{\Delta x} & \frac{\Delta v}{\Delta y} & \frac{\Delta v}{\Delta z} \\ \frac{\Delta w}{\Delta x} & \frac{\Delta w}{\Delta y} & \frac{\Delta w}{\Delta z} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

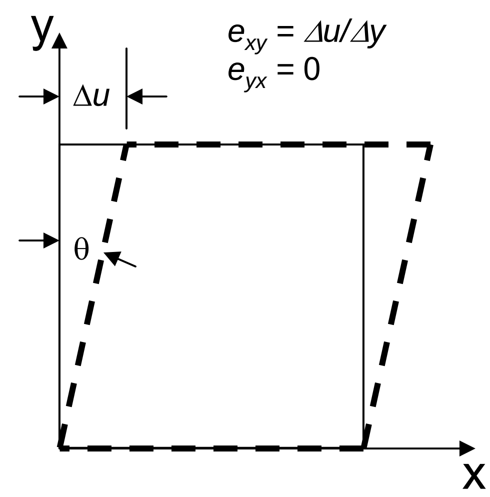
- The displacement strain matrix.
- Can produce pure shear strain and rigid-body rotation.



(1)
Pure Shear
w/o Rotation



(2)
Rotation



(3)
Simple Shear

- We need to break the displacement matrix into strain and rotational components.

- We can decompose the total strain matrix into symmetric and anti-symmetric components.

Decomposition of Strain

$$e_{ij} = \varepsilon_{ij} + \omega_{ij}$$

$$= \frac{1}{2}\left(e_{ij} + e_{ji}\right) + \frac{1}{2}\left(e_{ij} - e_{ji}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Symmetric | Anti-symmetric

Shear | Rotation

Displacement strain [matrix] ....

$$\begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix}$$

|| ||

Shear strain [tensor] ....

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} e_{xx} & \frac{1}{2}\left(e_{xy}+e_{yx}\right) & \frac{1}{2}\left(e_{xz}+e_{zx}\right) \\ \frac{1}{2}\left(e_{xy}+e_{yx}\right) & e_{yy} & \frac{1}{2}\left(e_{yz}+e_{zy}\right) \\ \frac{1}{2}\left(e_{xz}+e_{zx}\right) & \frac{1}{2}\left(e_{yz}+e_{zy}\right) & e_{zz} \end{bmatrix}$$

\+ \+

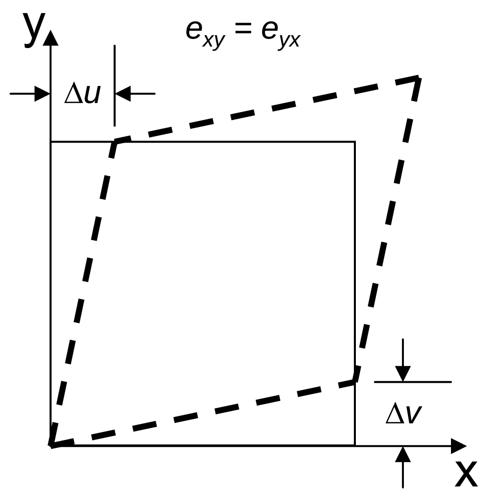
Rotation [tensor] ....

$$\omega_{ij} = \begin{bmatrix} \omega_{xx} & \omega_{xy} & \omega_{xz} \\ \omega_{xy} & \omega_{yy} & \omega_{yz} \\ \omega_{xz} & \omega_{yz} & \omega_{zz} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\left(e_{xy}-e_{yx}\right) & \frac{1}{2}\left(e_{xz}-e_{zx}\right) \\ \frac{1}{2}\left(e_{yx}-e_{xy}\right) & 0 & \frac{1}{2}\left(e_{yz}-e_{zy}\right) \\ \frac{1}{2}\left(e_{zx}-e_{xz}\right) & \frac{1}{2}\left(e_{zy}-e_{yz}\right) & 0 \end{bmatrix}$$

Shear Strain

- Total angular change from a right angle.

$$\gamma = e_{xy} + e_{yx} = 2\varepsilon_{xy} \quad (\omega_{ij} = 0)$$



(1)
Pure Shear
w/o Rotation

$$\gamma_{ij} = 2\varepsilon_{ij} \quad \text{(engineering shear strain)}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

Transformation of Strains

- Equations for strain, analogous to those for stress, can be written by substituting ε for σ and $\gamma/2$ for τ.

$$\sigma = \sigma_{\text{normal}} = \sigma_{xx} l^2 + \sigma_{yy} m^2 + \sigma_{zz} n^2 + \tau_{xy} 2lm + \tau_{yz} 2mn + \tau_{zx} 2nl$$

$$\downarrow$$

$$\varepsilon = \varepsilon_{\text{normal}} = \varepsilon_{xx} l^2 + \varepsilon_{yy} m^2 + \varepsilon_{zz} n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{zx} nl$$

- We can also define a coordinate system where there will be no shear strains. These will be principal axes

$$\varepsilon^3 - \left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)\varepsilon^2 + \left(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{xx}\varepsilon_{zz} - \frac{1}{4}\left(\gamma_{xy}^2 - \gamma_{yz}^2 - \gamma_{xz}^2\right)\right)\varepsilon$$

$$-\left(\varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz} + \frac{1}{4}\gamma_{xy}\gamma_{yz}\gamma_{xz} - \frac{1}{4}\left(\varepsilon_{xx}\gamma_{yz}^2 + \varepsilon_{yy}\gamma_{xz}^2 + \varepsilon_{zz}\gamma_{xy}^2\right)\right) = 0$$

or

$$\varepsilon^3 - I_1\varepsilon^2 + I_2\varepsilon - I_3 = 0$$

- The <u>directions in which the principal strains act</u> are determined by substituting ε_1, ε_2, and ε_3, each for ε in:

$$(\varepsilon_{xx} - \varepsilon)2l + \gamma_{yx}m + \gamma_{zx}n = 0$$
$$\gamma_{xy}l + (\varepsilon_{yy} - \varepsilon)2m + \gamma_{zy}n = 0$$
$$\gamma_{xz}l + \gamma_{yz}m + (\varepsilon_{zz} - \varepsilon)2n = 0$$

and then solving the resulting equations simultaneously for l, m, and n (using the relationship $l^2 + m^2 + n^2 = 1$).

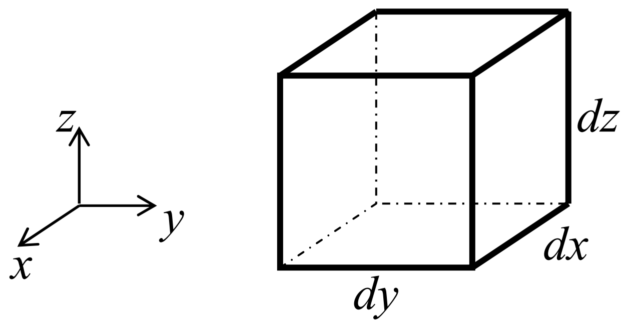
(a) Substitute ε_1 for ε in & solve; (b) Substitute ε_2 for ε in & solve; (c) Substitute ε_3 for ε in & solve.

Equations for Principal Shearing Strains

$$\gamma_1 = \varepsilon_2 - \varepsilon_3$$
$$\gamma_{\max} = \gamma_2 = \varepsilon_1 - \varepsilon_3$$
$$\gamma_3 = \varepsilon_1 - \varepsilon_2$$

- Deformation of a solid involves a combination of volume change and shape change.
- We can separate strain into hydrostatic (volume change) and deviatoric (shape change) components.

Hydrostatic Component



- Volume = *dxdydz*
- Volume of strained element
  $= (1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})dxdydz$

- The volume strain is:

$$\Delta = \frac{(1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})dxdydz - dxdydz}{dxdydz}$$

$$= (1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz}) - 1$$

- If we neglect the products of strains (i.e., $\varepsilon_{ii}\times\varepsilon_{jj}$), this becomes:

$$\Delta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

which is equal to the first invariant of the strain tensor

- The hydrostatic component of strain, i.e., the mean strain, is:

$$\varepsilon_{mean} = \frac{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}}{3} = \frac{\varepsilon_{ii}}{3} = \frac{\Delta}{3}$$

The mean strain does not induce shape change. It causes volume change. It is the hydrostatic component.

- The part that causes shape change is called the strain deviator. We get the strain deviator by subtracting the mean strain from the normal strain components.

$$\varepsilon'_{ij} = \begin{vmatrix} \varepsilon_{xx} - \varepsilon_{mean} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} - \varepsilon_{mean} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} - \varepsilon_{mean} \end{vmatrix}$$

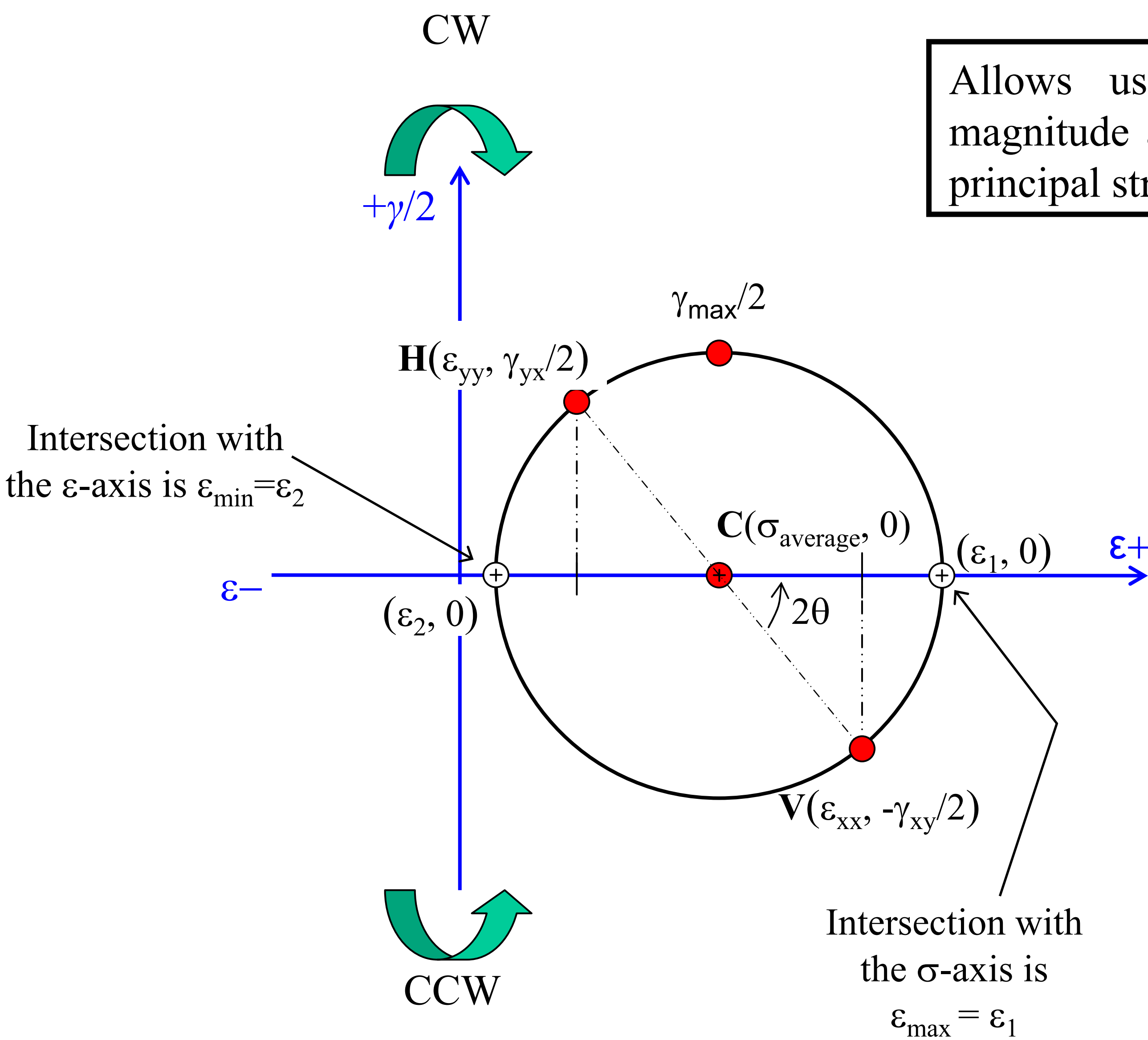
The Strain Deviator

$$\varepsilon'_{ij} = \begin{vmatrix} \varepsilon_{xx} - \varepsilon_{mean} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} - \varepsilon_{mean} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} - \varepsilon_{mean} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{2\varepsilon_{xx} - \varepsilon_{yy} - \varepsilon_{zz}}{3} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \dfrac{2\varepsilon_{yy} - \varepsilon_{zz} - \varepsilon_{xx}}{3} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \dfrac{2\varepsilon_{zz} - \varepsilon_{xx} - \varepsilon_{yy}}{3} \end{vmatrix}$$

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon_m = \left(\varepsilon_{ij} - \frac{\Delta}{3}\delta_{ij} \right) + \frac{\Delta}{3}\delta_{ij}$$

Mohr's Circle for Strain



Allows us to determine the magnitude and directions of the principal strains.

MAXIMUM & MINUMUM PRINCIPAL STRAINS IN 2-D STATE

$$\begin{matrix}\varepsilon_{\max} \\ \varepsilon_{\min}\end{matrix} = \begin{matrix}\varepsilon_1 \\ \varepsilon_2\end{matrix} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

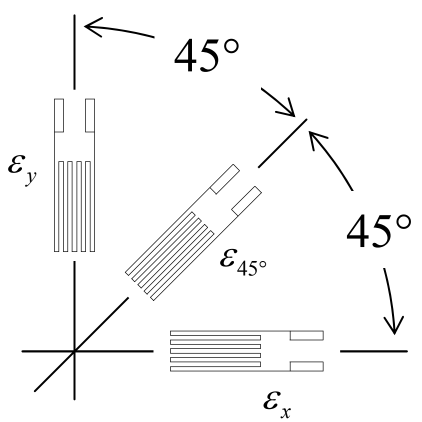
$$\gamma_{\max} = \gamma_3 = \sqrt{\left(\varepsilon_{xx} - \varepsilon_{yy}\right)^2 + \left(\gamma_{xy}\right)^2}$$

$$\tan 2\theta_{normal} = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}}$$

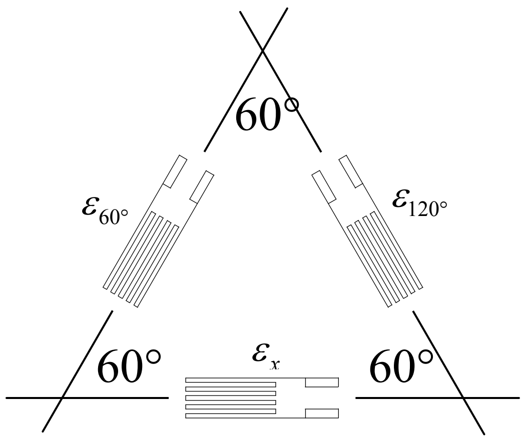
$$\tan 2\theta_{shear} = -\frac{\varepsilon_{xx} - \varepsilon_{yy}}{\gamma_{xy}}$$

Strain Measurement

- Strain can be measured using a strain gauge.
- When an object is deformed, the wires in the strain gate are strained which changes their electrical resistance, which is proportional to strain.
- Strain gauges make only direct readings of linear strain. Shear strain must be determined indirectly.



Rectangular



Delta

State of stress at a point:

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

$$\equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$

State of strain at a point:

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \equiv \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix}$$

There are many different systems of notation.
BE WARY!

Matrix Notation

- We often replace the indices with matrix notation for simplicity

$$xx \rightarrow 11 \rightarrow 1 \quad yy \rightarrow 11 \rightarrow 2 \quad zz \rightarrow 33 \rightarrow 3$$
$$yz \rightarrow 23 \rightarrow 4 \quad xz \rightarrow 13 \rightarrow 5 \quad xy \rightarrow 12 \rightarrow 6$$

$$\begin{pmatrix} 11 & & 12 & \leftarrow & 13 \\ & \searrow & & & \uparrow \\ & & 22 & & 23 \\ & & & \searrow & \uparrow \\ & & & & 33 \end{pmatrix} \equiv \begin{pmatrix} 1 & & 6 & \leftarrow & 5 \\ & \searrow & & & \uparrow \\ & & 2 & & 4 \\ & & & \searrow & \uparrow \\ & & & & 3 \end{pmatrix}$$

- This will be particularly important when we discuss higher order tensors and tensor relationships (i.e., elastic properties)

General forms for stress and strain in matrix notation

$$\begin{pmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_1 & \frac{\varepsilon_6}{2} & \frac{\varepsilon_5}{2} \\ \frac{\varepsilon_6}{2} & \varepsilon_2 & \frac{\varepsilon_4}{2} \\ \frac{\varepsilon_5}{2} & \frac{\varepsilon_5}{2} & \varepsilon_3 \end{pmatrix}$$

NOTE

$$\varepsilon_1 = \varepsilon_{11}; \varepsilon_2 = \varepsilon_{22}; \varepsilon_3 = \varepsilon_{33}$$

Special definitions →

$$\varepsilon_4 = 2\varepsilon_{23} = \gamma_{23}$$

$$\varepsilon_5 = 2\varepsilon_{13} = \gamma_{13}$$

$$\varepsilon_6 = 2\varepsilon_{12} = \gamma_{12}$$