Module #6b

Yield/failure criteria

READING LIST

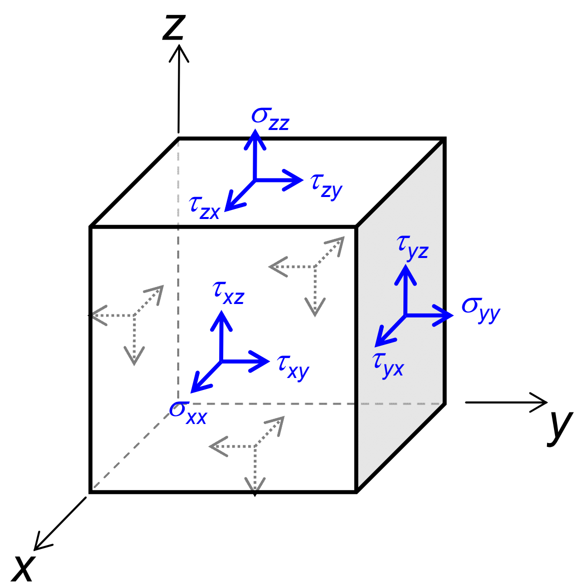
DIETER: Ch. 2, pp. 36-38; Ch. 3, pp. 77-85
Dowling: Ch. 7, pp. 254-302

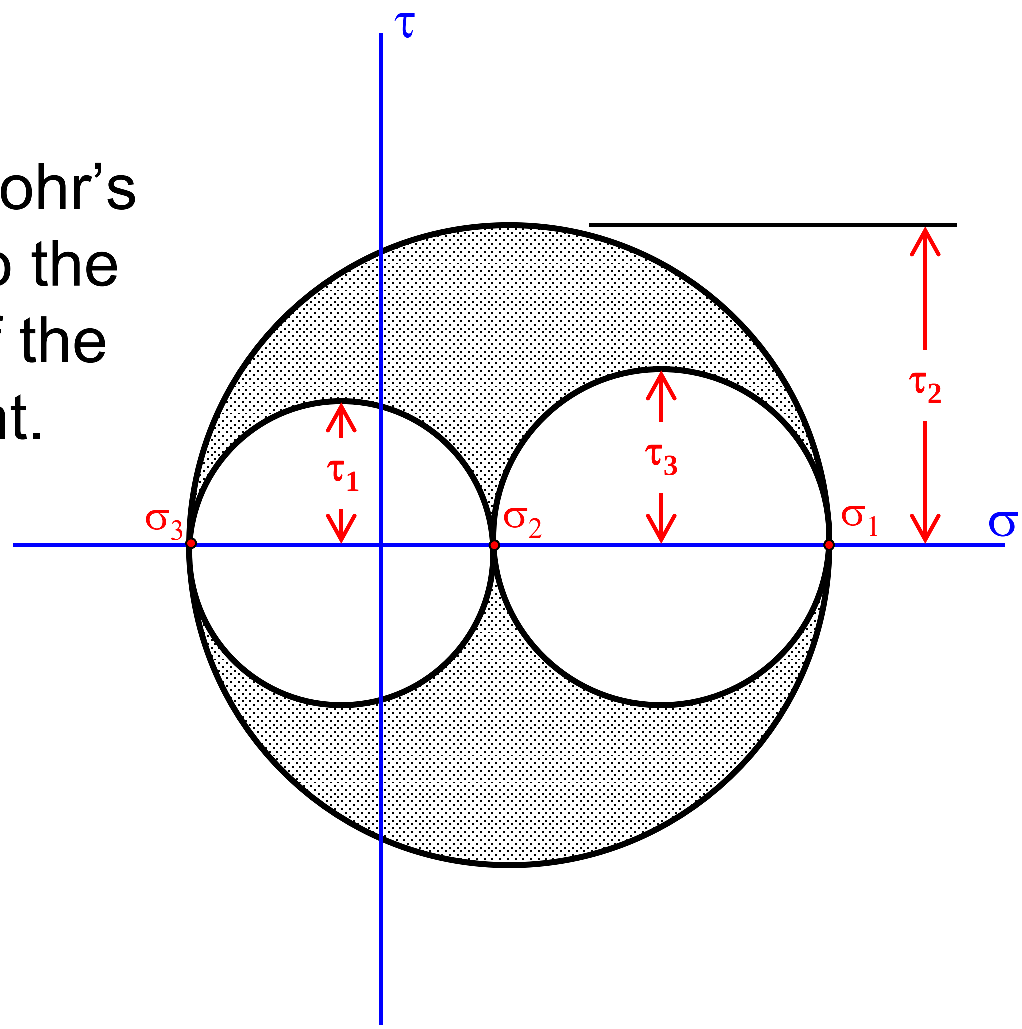
Ch. 3 in Roesler
Ch. 2 in McClintock and Argon
Ch. 7 in Edelglass

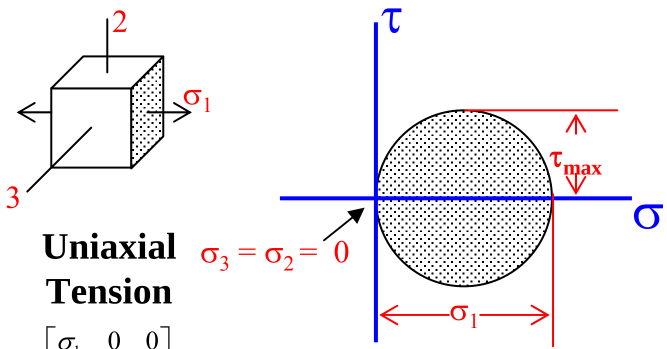
Mohr's Circle in 3-D

- We can use a 3-D Mohr's circle to visualize the state of stress and to determine principal stresses.

- Essentially three 2-D Mohr's circles corresponding to the *x-y, x-z, and y-z* faces of the elemental cubic element.

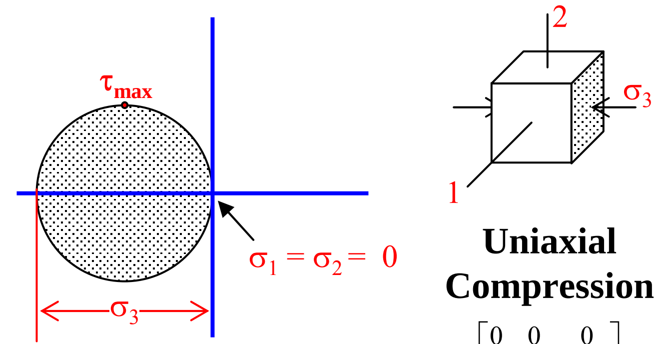






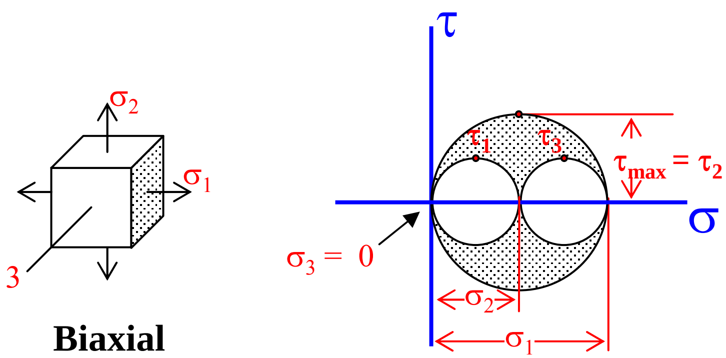
Uniaxial Tension

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



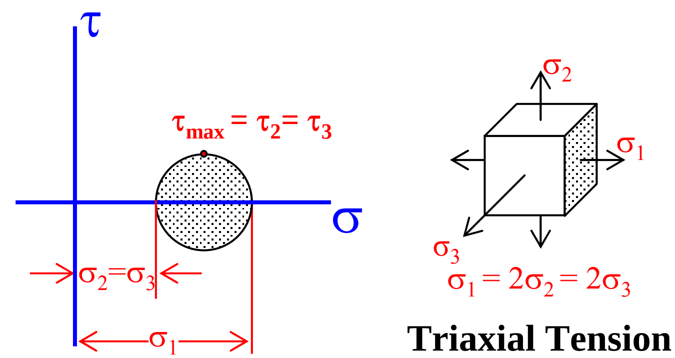
Uniaxial Compression

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sigma_3 \end{bmatrix}$$



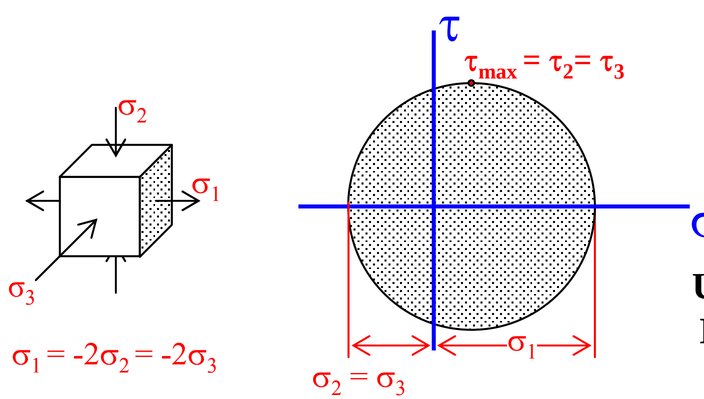
Biaxial Tension

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Triaxial Tension (unequal)

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$



Uniaxial Tension plus Biaxial Compression

Adapted from G.E. Dieter, Mechanical Metallurgy, 3rd ed., McGraw-Hill (1986) p. 37

Influence of States of Stress

- Biaxial and triaxial tension:
  - Effectively reduces the shear stresses resulting in a considerable decrease in ductility. (Plastic deformation is produced by shear stresses.)
- Uniaxial tension plus biaxial compression:
  - Produces high shear stresses and contributes towards increased plastic deformation without fracture.
  - This is like metal forming via extrusion which gives better ductility than uniaxial tension.

Multiaxial Loading

Most service conditions and forming operations (Ex., drawing) involve multiaxial loading.

Under multiaxial loading conditions, a material or structure may yield or fracture locally (or globally) depending upon the state of stress.

We can use the calculated principal stresses to define criteria for yielding or failure.

Yield/Failure Criteria (1)

Mathematical tools to decide whether the stress state in a material will cause plastic deformation or failure.

Consider an isotropic polycrystalline metal deformed in uniaxial tension. It will yield when:

$$\sigma_{applied} = \sigma_{YS}$$

This is a valid yield criterion for the stated problem.

Yield/Failure Criteria (2)

Consider the same isotropic polycrystalline metal deformed in a multiaxial stress state.

We can’t simply determine the stress at yielding because stress will vary from point to point.

Instead we calculate an equivalent stress from the components of the stress tensor and compare it with the critical stress for yielding/failure.

Yield/Failure Criteria (3)

The equivalent stress, being a function of the stress tensor, can be expressed as:

$$\sigma_{equivalent}\left(\sigma_{xx},\sigma_{yy},\sigma_{zz},\tau_{yz},\tau_{xz},\tau_{xy}\right)$$

At yielding/failure, this equivalent stress much reach the critical value (e.g., σ_{YS}, σ_f, τ_{CRSS}, etc.). Thus:

$$\sigma_{equivalent}\left(\sigma_{xx},\sigma_{yy},\sigma_{zz},\tau_{yz},\tau_{xz},\tau_{xy}\right)=\sigma_{critical}$$

$$or$$

$$\sigma_{equivalent}\left(\sigma_{ij}\right)=\sigma_{critical}$$

(at failure)

Yield/Failure Criteria (4)

Yield criteria are generally expressed as:

$$f\left(\sigma_{ij}\right) = \sigma_{equivalent}\left(\sigma_{ij}\right) - \sigma_{critical} = 0$$

Thus, when $f(\sigma_{ij}) < 0$, the material does not yield/fail.

When $f(\sigma_{ij}) \geq 0$, the material yields/fails.

Example:

Uniaxial tension. Material deforms elastically up to the yield stress. When applied load reaches the critical load (i.e., the YS), plastic deformation occurs. The yield/failure criterion could be expressed as:

$$f\left(\sigma_{ij}\right) = \sigma_{applied} - \sigma_{YS} = 0$$

Yield/Failure Criteria (5)

For isotropic materials, we can express yield criteria in terms of principal stresses.

$$f\left(\sigma_1,\sigma_2,\sigma_3\right)=0$$

If we plot the function $f(\sigma_1,\sigma_2,\sigma_3)$ on orthogonal σ_1, σ_2, σ_3 axes we obtain a yield surface.

We can use the yield surface to determine, for each possible state of stress, whether or not a material yields/fails.

Yield/Failure Criteria (6)

There are many different yield criteria.

We will limit ourselves to these three:

1. Rankine
2. Tresca
3. von Mises

Remember, there are more than these two.

Yield (Failure) Criteria (7)

Tresca (Maximum-Shear-Stress) Yield Criterion:

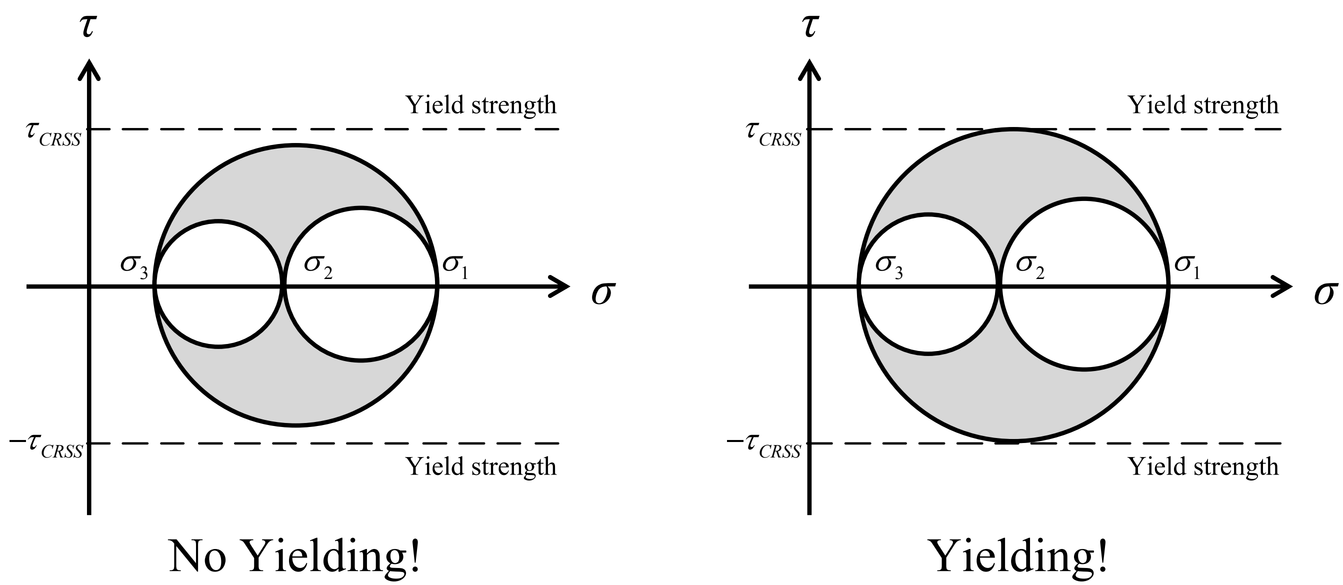
- Yielding occurs when the difference between the maximum and minimum normal stresses reaches a critical value, the yield strength

$$\sigma_{max} - \sigma_{min} = \sigma_1 - \sigma_3 = \sigma_{ys}$$

[for a tensile test]

OR

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{ys}}{2} = \tau_{CRSS}$$



Yield (Failure) Criteria (8)

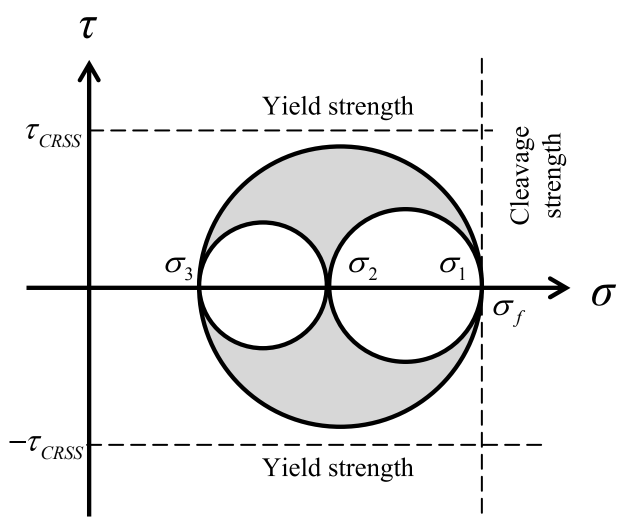
Rankine (Maximum-Principal-Stress) Criterion:

- Cleavage fracture occurs when the cleavage strength is reached before the yield strength.

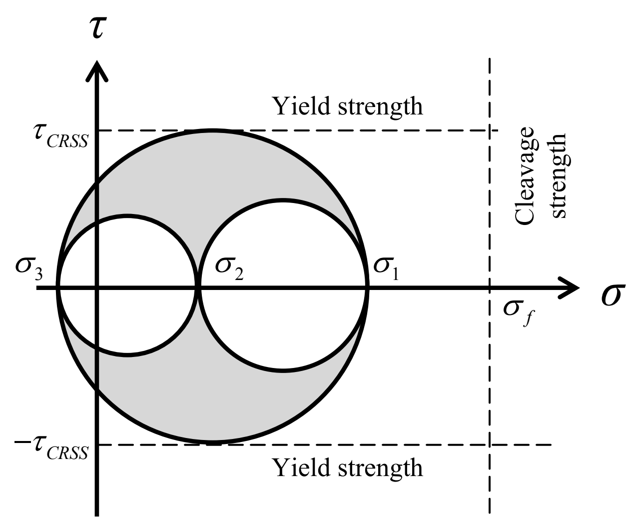
$$\sigma_{eq}\left(\sigma_1, \sigma_2, \sigma_3\right) < \sigma_{ys}$$

BUT

$\sigma_1 \geq \ \sigma_f$ (the cleavage strength)



Cleavage fracture!



Yielding!

Yield (Failure) Criteria (9)

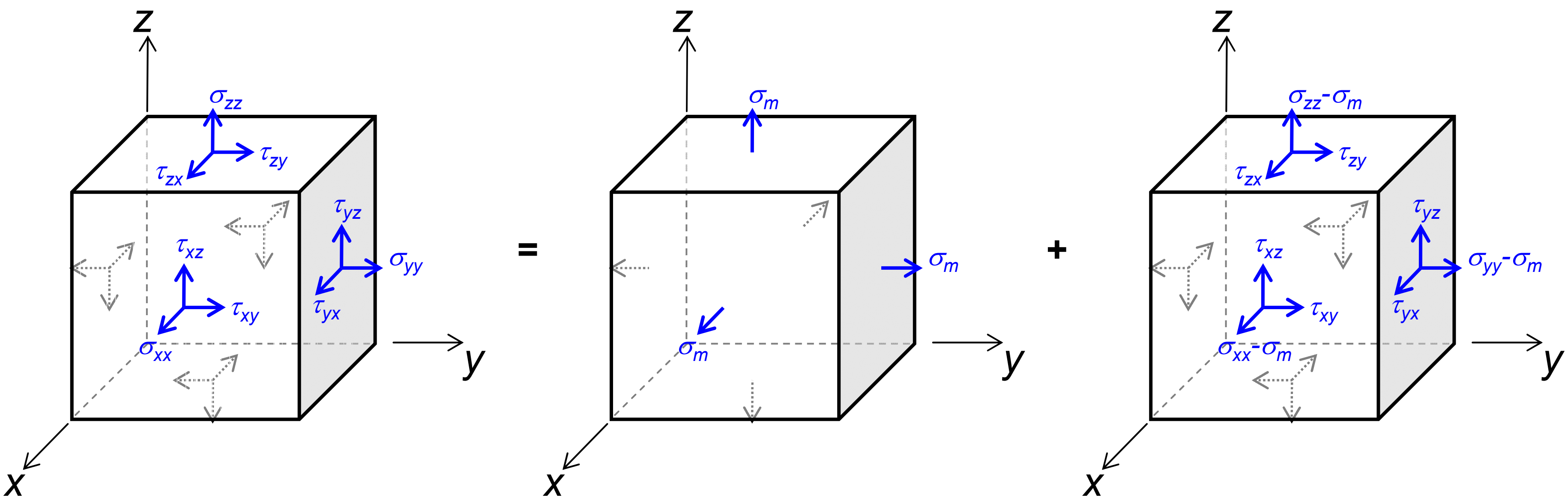
Von-Mises Yield Criterion (Distortion Energy Criterion):

- Yielding occurs when the second invariant of the stress deviator, J_2, exceeds a critical value or:

$$J_2 = \text{constant} = k^2.$$

- What is the stress deviator and how do I find k^2?

- The stress deviator represents the part of the total stress state that causes shape change (i.e., deformation).



TOTAL

HYDROSTATIC [ISOTROPIC]

DEVIATOR

causes dilation

Volume change

No shape change!

causes distortion

Shape change

The Stress Deviator (1)

$$\underbrace{\begin{vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{vmatrix}}_{\text{Total Stress}} = \underbrace{\begin{vmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{vmatrix}}_{\text{Hydrostatic Stress}} + \underbrace{\begin{vmatrix} (\sigma_{xx} - \sigma_m) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma_m) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma_m) \end{vmatrix}}_{\text{Stress Deviator}}$$

The hydrostatic stress, σ_m, does not cause plastic deformation

$$\sigma_m = P = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{I_1}{3}$$

The stress deviatior causes plastic deformation

The Stress Deviator (2)

$$\sigma'_{ij} = \begin{vmatrix} (\sigma_{xx} - \sigma_m) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma_m) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma_m) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{2\sigma_{xx} - \sigma_{yy} - \sigma_{zz}}{3} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \frac{2\sigma_{yy} - \sigma_{zz} - \sigma_{xx}}{3} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \frac{2\sigma_{zz} - \sigma_{xx} - \sigma_{yy}}{3} \end{vmatrix}$$

- Take the **determinant of the stress deviator**.

The Stress Deviator (3)

- This yields a new cubic equation that has three new invariants:

$$\sigma'^3 - J_1\sigma'^2 + J_2\sigma' - J_3 = 0$$

- The invariants are the:
  (1) sum of main diagonal;
  (2) sum of principal minors;
  (3) determinant of deviator tensor.

The Stress Deviator (4)

- Two of the new invariants, the invariants of the stress deviator, are of great importance:

$$J_1 = I_1 - \sigma_m = (\sigma_{xx} - \sigma_m) + (\sigma_{yy} - \sigma_m) + (\sigma_{zz} - \sigma_m)$$

$$J_2 = I_2 - \sigma_m = \frac{1}{6}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)\right]$$

$$= \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

The Stress Deviator (5)

- In uniaxial tension, yielding occurs when $\sigma_1 = \sigma_{YS}$ (yield stress) and $\sigma_2 = \sigma_3 = 0$. Thus J_2 becomes:

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$= \frac{1}{6}\left[(\sigma_{YS})^2 + (-\sigma_{YS})^2\right]$$

$$= k^2$$

$$= \frac{\sigma_{YS}^2}{3}$$

- It represents the condition required to cause yielding.

The Stress Deviator (6)

- Therefore, the von Mises criterion becomes:

$$\frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2}=3k^2=\sigma_{YS}=YS$$

- Yielding occurs when J_2 equals or exceeds the tensile yield stress.

- Look at Figure 1.11 from Courtney's text (next page). This figure shows the yield locus for **plane stress**.

- **States of stress with principal stresses lying within the bounds of the yield locus do not produce yielding.**

Quadrant I: yielding occurs when σ_1 or $\sigma_2 = \sigma_{ys}$

$\sigma_{max}\ (\sigma_1) - \sigma_{min}\ (\sigma_3 = 0) = \sigma_{ys}$

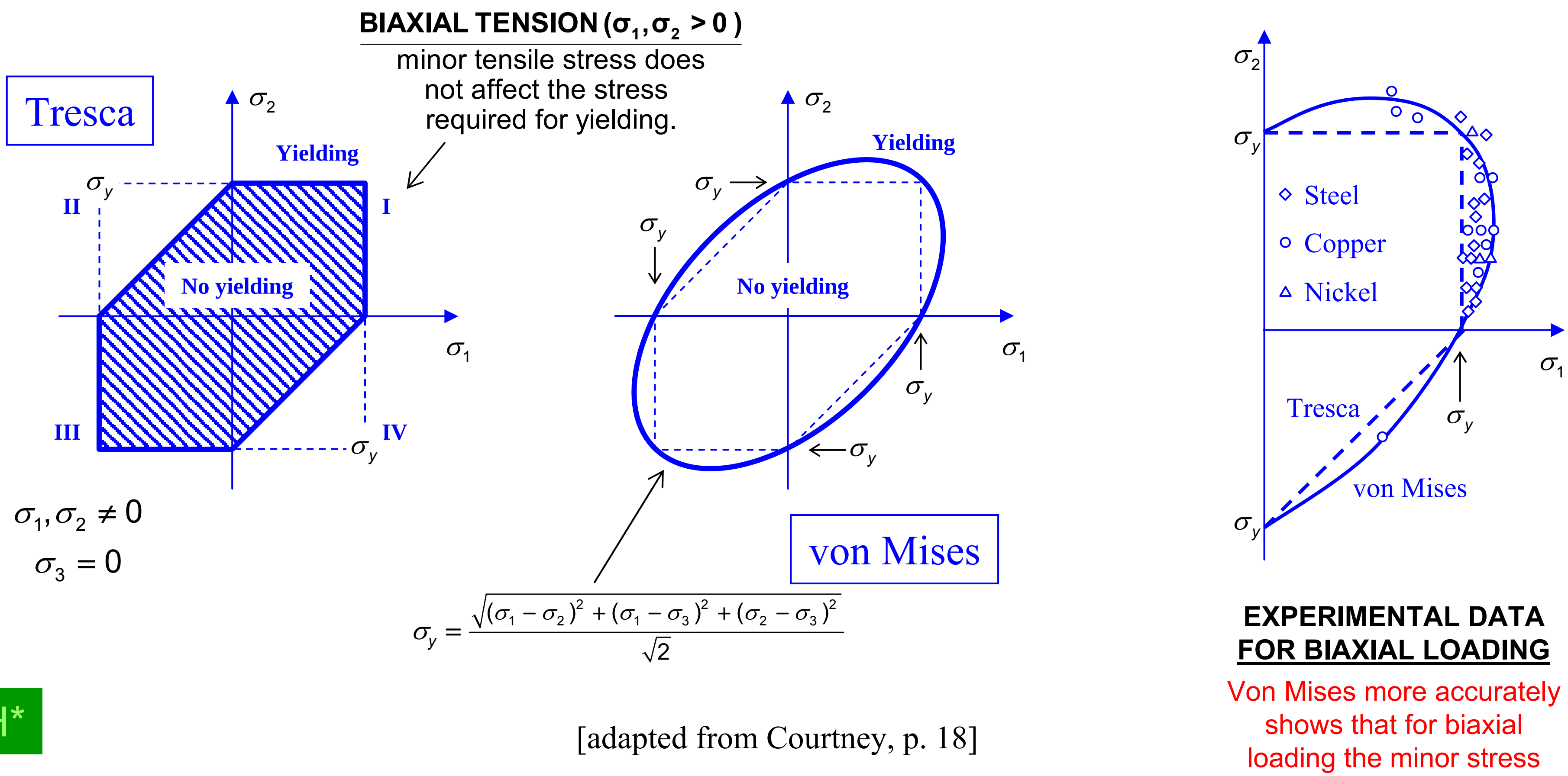
Quadrant II: $\sigma_1\ (\sigma_{min}) < 0;\ \sigma_2 \geq 0;\ \sigma_3 = 0)$

$\sigma_2 - \sigma_1 = \sigma_{ys}$

so neither $|\sigma_1|$ or $|\sigma_2| = \sigma_{ys}$ and still get yielding

- The Tresca criterion is less complicated (in terms of math). It is often used in engineering design. It is also more conservative.

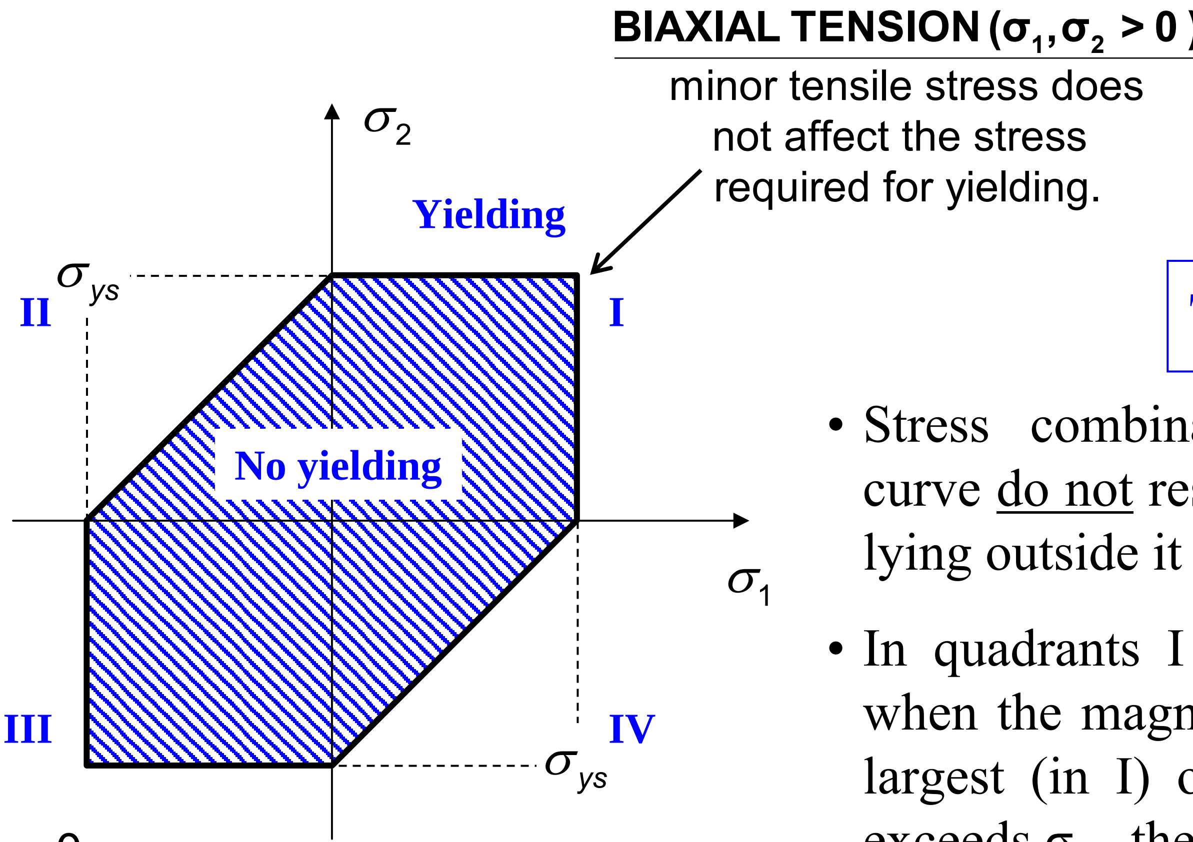
- However, the Tresca criterion does not take into account the intermediate principal stress and requires that you know the maximum and minimum principal stresses.



H*

FIGURE 1.11

(a) The Tresca yield condition for biaxial loading. Stress combinations lying within the curve do not result in plastic flow; those lying outside it do. In quadrants I and III, yielding occurs when the magnitude of the algebraically largest (in I) or smallest (in III) stress exceeds σ_{ys}, the tensile yield strength. In quadrant II, (σ_2>0, σ_1<0, σ_3=0), yielding is defined by σ_{max} (=σ_2)-σ_{min}(=σ_1) =σ_{ys}, and this results in a 45° line defining yielding. The yield criterion is similar in quadrant IV (σ_1>0, σ_2<0, σ_3 = 0), except that σ_1 and σ_2 are interchanged. (b) The von Mises yield condition for biaxial loading is shown by the solid line. Stress combinations lying within the ellipse do not lead to plastic flow; those lying outside do. The Tresca condition (dotted line) is compared to the von Mises one in the figure. The former is more conservative and the two are equivalent only for uniaxial ($\sigma_{1,2}$>0 with $\sigma_{2,1}$=σ_3=0), and balanced biaxial (σ_1=σ_2, σ_3=0), tension. (c) Comparison of experimental data for selected metals with the Tresca and von Mises criteria. The latter clearly fits the better data, though the difference between the criteria is not great.



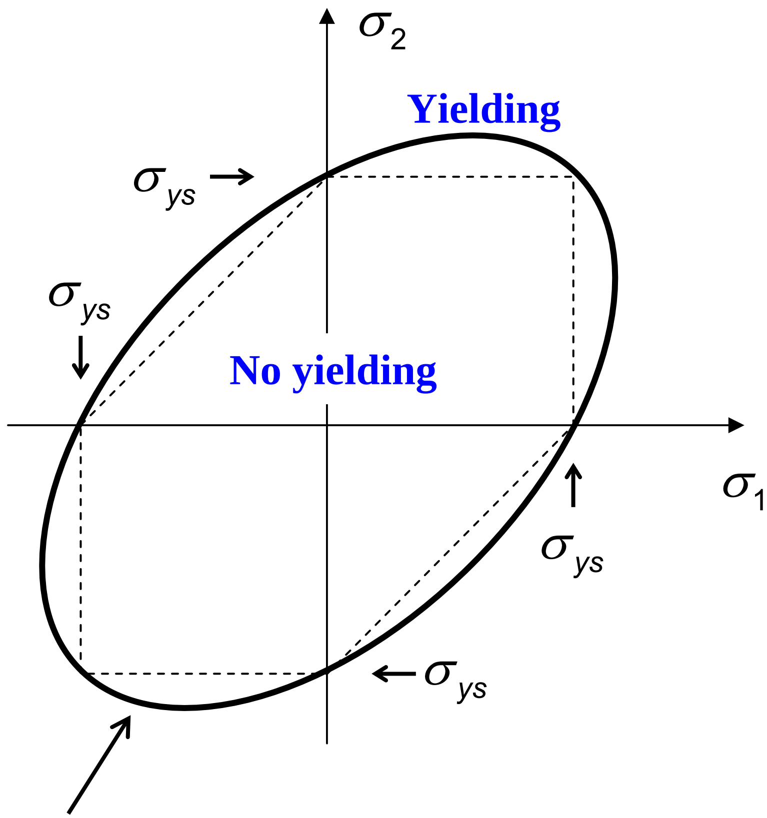
$\sigma_1, \sigma_2 \neq 0$

$\sigma_3 = 0$

[adapted from Courtney, p. 18]

Tresca

- Stress combinations lying within the curve do not result in plastic flow; those lying outside it do.
- In quadrants I and III, yielding occurs when the magnitude of the algebraically largest (in I) or smallest (in III) stress exceeds σ_{ys}, the tensile yield strength.
- In quadrant II, ($\sigma_2>0$, $\sigma_1<0$, $\sigma_3=0$), yielding is defined by σ_{max} ($=\sigma_2$)-σ_{min}($=\sigma_1$) $=\sigma_{ys}$, and this results in a 45° line defining yielding. The yield criterion is similar in quadrant IV ($\sigma_1>0$, $\sigma_2<0$, $\sigma_3 = 0$), except that σ_1 and σ_2 are interchanged.

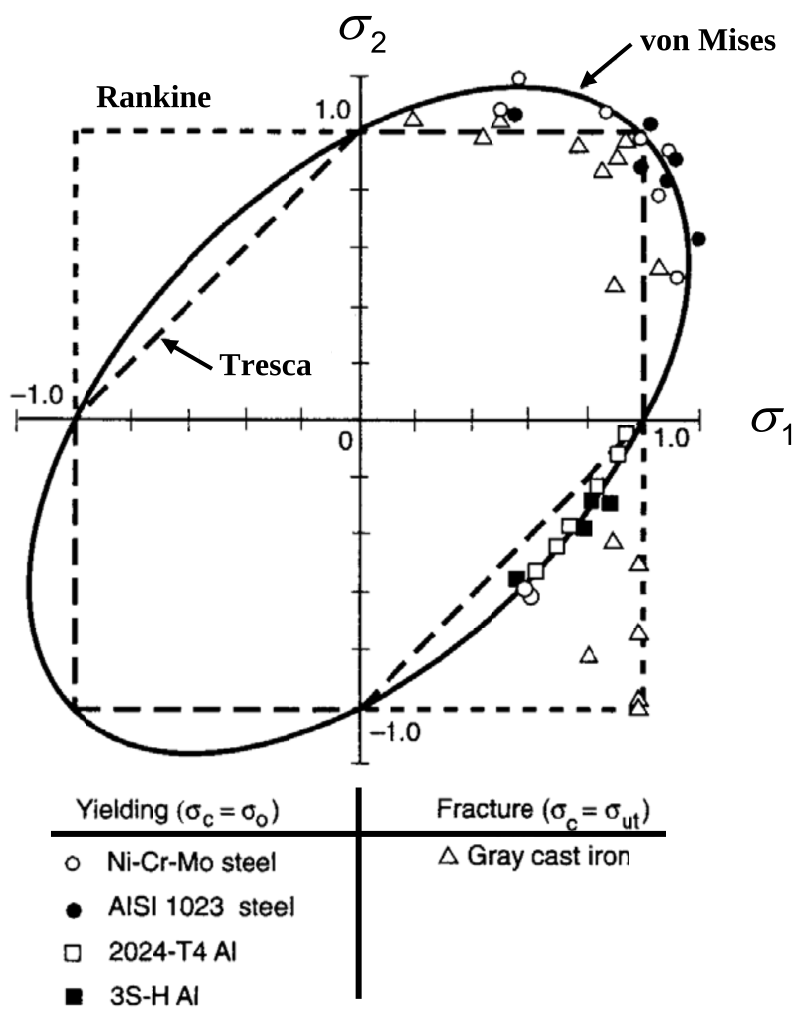


$$\sigma_{ys} = \frac{\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}}{\sqrt{2}}$$

[adapted from Courtney, p. 18]

von Mises

- The von Mises yield condition for biaxial loading is shown by the solid line.
- Stress combinations lying within the ellipse do not lead to plastic flow; those lying outside do.
- The Tresca condition (dotted line) is compared to the von Mises one in this figure.
- The former (Tresca) is more conservative and the two are equivalent only for uniaxial ($\sigma_1, \sigma_2 > 0$ with $\sigma_2, \sigma_1 = \sigma_3 = 0$), and balanced biaxial ($\sigma_1 = \sigma_2, \sigma_3 = 0$), tension.



From, Mechanical Behavior of Materials: Engineering Methods for Deformation, Fracture, and Fatigue, Third Edition, p. 275, by Norman E. Dowling. ISBN 0-13-186312-6.

Comparison

- Comparison of experimental data for selected metals with the Tresca, Rankine, and von Mises criteria.
- The latter (von Mises) clearly fits the data better for ductile metals.
- The Rankine criterion fits a brittle metal like gray cast iron quite well.
- However, the difference between the criteria is not great.

Example Problem

- The yield strength for a new Ni-base superalloy is 1000 MPa. Determine whether yielding will have occurred on the basis of both the Tresca and Von Mises failure criteria assuming the following stress state.

$$\begin{bmatrix} 0 & 0 & 500 \\ 0 & -200 & 0 \\ 500 & 0 & -900 \end{bmatrix} \text{MPa}$$

Example Problem

1. First determine the principal stresses

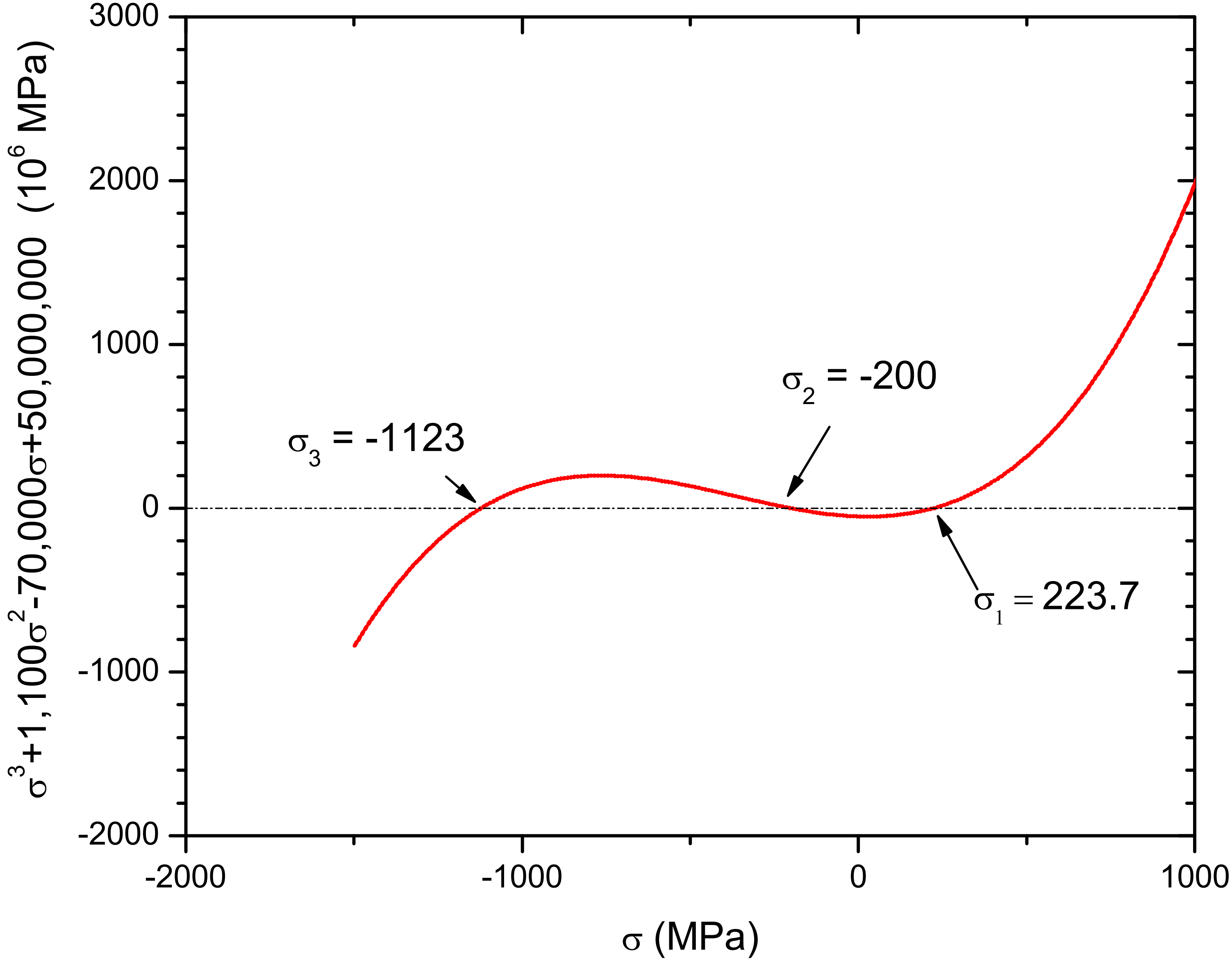
$$\begin{bmatrix} 0 & 0 & 500 \\ 0 & -200 & 0 \\ 500 & 0 & -900 \end{bmatrix} \text{MPa}$$

$$\sigma^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)\sigma$$
$$-(\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{xz}^2 - \sigma_{zz}\tau_{xy}^2) = 0$$

$$\sigma^3 - (0 - 200 - 900)\sigma^2 + [(0 \cdot -200) + (-200 \cdot -900) + (0 \cdot -900) - (0)^2 - (500)^2 - (0)^2]\sigma$$
$$-[(0 \cdot -200 \cdot -900) + 2(0 \cdot 500 \cdot 0) - (0 \cdot 0^2) - (-200 \cdot 500^2) - (-900 \cdot 0^2)] = 0$$

$$\sigma^3 - (-1100)\sigma^2 + [-70,000]\sigma - [-50,000,000] = 0$$

$$\sigma^3 + 1,100\sigma^2 - 70,000\sigma + 50,000,000 = 0$$



3000
2000
1000
0
-1000
-2000
-2000
-1000
0
1000
σ (MPa)
σ³+1,100σ²-70,000σ+50,000,000 (10⁶ MPa)
σ₃ = -1123
σ₂ = -200
σ₁ = 223.7

2. Substitute principal stresses into equations for Tresca and Von Mises failure criteria

- Principal stresses:

σ_1 = 223.7; σ_2 = -200.0; σ_3 = -1123.0

- Von Mises:

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$
$$= \frac{1}{6}\left[(223.7 - (-200))^2 + ((-200) - (1123))^2 + ((-1123) - 223.7)^2\right]$$
$$= 623,908.6$$

For yielding to occur $J_2 \geq k^2$ $(=\sigma_o^2/3)$ therefore k = 789.9.

$\sigma_o = k \times (3)^{1/2} = 1368.1$ MPa

Since 1168.1 MPa > 1000 MPa, yielding occurs.

- Tresca:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{223.7 - (-1123.0)}{2} = 673.4 = \frac{\sigma_o}{2} \quad \underline{\text{OR}} \quad \sigma_o = 2 \times 673.4 = 1346.7$$

Since σ_o = 1346.7 MPa > 1000 MPa, yielding occurs.